C_λ-Extended Oscillator Algebras and Some of Their Deformations and Applications to Quantum Mechanics

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C_λ-extended oscillator algebras generalizing the Calogero–Vasiliev algebra, where C_λ is the cyclic group of order λ, are studied both from mathematical and applied viewpoints. Casimir operators of the algebras are obtained and used to provide a complete classification of their unitary irreducible representations under the assumption that the number operator spectrum is nondegenerate. Deformed algebras admitting Casimir operators analogous to those of their undeformed counterparts are looked for, yielding three new algebraic structures. One of them includes the Brzeziński *et al.* deformation of the Calogero–Vasiliev algebra as a special case. In its bosonic Fock-space representation, the realization of C_λ-extended oscillator algebras as generalized deformed oscillator ones is shown to provide a bosonization of several variants of supersymmetric quantum mechanics: parasupersymmetric quantum mechanics of order $p = \lambda - 1$ for any λ, as well as pseudosupersymmetric and orthosupersymmetric quantum mechanics of order two for $\lambda = 3$.

1. INTRODUCTION

The oscillator algebra of creation, annihilation, and number operators plays a central role in the investigation of many physical systems and provides a useful tool in the theory of Lie algebra representations. Similarly, some of its deformations (or extensions) have found applications to various physical problems, such as the description of systems with nonstandard statistics (Greenberg, 1990, 1991; Fivel, 1990; Meljanac *et al.*, 1994; Meljanac and Mileković, 1996; Quesne, 1994a), the construction of integrable lattice models (Bogoliubov *et al.*, 1994), the investigation of nonlinearities in quantum

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optics (McDermott and Solomon, 1994; Solomon, 1998; Man'ko *et al.*, 1997), the bosonization of supersymmetric quantum mechanics (SSQM) (Bonatsos and Daskaloyannis, 1993a; Brzeziński *et al.*, 1993; Plyushchay, 1996a, b; Beckers *et al.*, 1997), as well as the algebraic treatment of quantum exactly solvable models (Daskaloyannis, 1992; Bonatsos and Daskaloyannis, 1993b; Bonatsos *et al.*, 1993, 1994; Quesne, 1994b), n-particle integrable systems (Vasiliev, 1991; Polychronakos, 1992; Brink *et al.*, 1992; Brink and Vasiliev, 1993; Quesne, 1995), pairing correlations in nuclei (Bonatsos, 1992; Bonatsos and Daskaloyannis, 1992a), and vibrational spectra of molecules (Chang *et al.*, 1991; Chang and Yan, 1991a–c; Bonatsos and Daskaloyannis, 1992b, 1993c). In addition, they have been used to construct representations of quantum universal enveloping algebras of Lie algebras, also referred to as quantum algebras (Biedenharn, 1989; Macfarlane, 1989; Sun and Fu, 1989; Hayashi, 1990; Fairlie and Zachos, 1991; Fairlie and Nuyts, 1994).

Deformations of the oscillator algebra arose from successive generalizations of the Arik-Coon (Arik and Coon, 1976; Kuryshkin, 1980) and Biedenharn–Macfarlane (Biedenharn, 1989; Macfarlane, 1989; Sun and Fu, 1989) q-oscillators. Various attempts have been made to introduce some order in the rich and varied choice of deformed commutation relations by defining 'generalized deformed oscillator algebras' (GDOAs). Among them, one may quote the treatments due to Jannussis *et al.* (1991), Jannussis (1993), Daskaloyannis (1991), Bonatsos and Daskaloyannis (1993a), Irac-Astaud and Rideau (1992, 1993, 1994), McDermott and Solomon (1994), Meljanac *et al.* (1994), Meljanac and Mileković (1996), Katriel and Quesne (1996), and Quesne and Vansteenkiste (1995, 1996, 1997). In the remainder of the present paper, we shall refer to GDOAs as defined in the last references.

The G-extended oscillator (or alternatively Heisenberg[3]) algebras, where G is some finite group, appeared in connection with n-particle integrable models. It was shown (Vasiliev, 1991; Polychronakos, 1992; Brink *et al.*, 1992; Brink and Vasiliev, 1993; Quesne, 1995) that they provide an algebraic formulation of the Calogero model (Calogero, 1969a, b, 1971), or some generalization thereof (Wolfes, 1974; Calogero and Marchioro, 1974). In the former case, G is the symmetric group S_n (Polychronakos, 1992; Brink *et al.*, 1992; Brink and Vasiliev, 1993). For two particles, the Abelian group S_2 can be realized in terms of Klein operator $K = (-1)^N$, where N denotes the number operator. The S_2-extended oscillator algebra then becomes a GDOA, also known as the Calogero–Vasiliev (Vasiliev, 1991) or modified (Brzeziński *et al.*, 1993) oscillator algebra. Some deformations of the latter have been

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[3] In both the oscillator and Heisenberg algebras, the creation and annihilation operators $a^\dagger$, a are considered as generators, but in the former the number operator N appears as an additional independent generator, whereas in the latter it is defined in terms of $a^\dagger$, a as $N \equiv a^\dagger a$.

extensively studied (Brzeziński *et al.*, 1993; Macfarlane, 1994; Kosiński *et al.*, 1997; Tsohantjis *et al.*, 1997; Paolucci and Tsohantjis, 1997).

The purpose of the present paper is to study a new class of *G*-extended oscillator algebras (Quesne and Vansteenkiste, 1998), generalizing the one describing the two-particle Calogero model. Here *G* is the cyclic group of order λ, $C_\lambda = \{I, T, T^2, \ldots, T^{\lambda-1}\}$, which for $\lambda = 2$ is isomorphic to S_2. Such C_λ-extended oscillator algebras $\mathcal{A}^{(\lambda)}_{\alpha_0\alpha_1\ldots\alpha_{\lambda-2}}$ have a rich structure, since they depend upon $\lambda - 1$ independent real parameters α_0, α_1, $\ldots$, $\alpha_{\lambda-2}$ (reducing to a single one in the $\lambda = 2$ case, corresponding to the S_2-extended oscillator algebra). Realizing *T* in terms of the number operator *N* converts $\mathcal{A}^{(\lambda)}_{\alpha_0\alpha_1\ldots\alpha_{\lambda-2}}$ into a GDOA $\mathcal{A}^{(\lambda)}(G(N))$.

The bosonic oscillator Hamiltonian H_0 associated with $\mathcal{A}^{(\lambda)}(G(N))$ is equivalent to the two-particle Calogero Hamiltonian for $\lambda = 2$, but exhibits entirely new features for $\lambda \geq 3$ (Quesne and Vansteenkiste, 1998). In such a case, all the levels corresponding to a number of quanta equal to μ mod λ are equally spaced, but the ordering and spacing of levels associated with different μ values depend on the algebra parameters α_0, α_1, $\ldots$, $\alpha_{\lambda-2}$. By appropriately choosing the latter, one may therefore obtain nondegenerate spectra, as well as spectra exhibiting some $(\nu + 1)$-fold degeneracies, where ν may take any value in the set $\{1, 2, \ldots, \lambda - 1\}$.

The rich variety of spectra that may be obtained with H_0, as well as the connection with the Calogero model for $\lambda = 2$, makes it most likely that some interesting applications will arise in one or another context. To help toward finding them, the construction of realizations of the $\mathcal{A}^{(\lambda)}(G(N))$ generators in terms of differential operators is under current investigation and will be reported elsewhere.

We may already note, however, that spectra that are a strictly equidistant continuation of a triplet of 'ground' states (which can be obtained here for $\lambda = 3$) arose in two studies of a class of potentials (with applications in string theory) using either an advanced factorization method (Veselov and Shabat, 1993) or a nonlinear generalization of the Fock method (Eleonsky *et al.*, 1994, 1995; Eleonsky and Korolev, 1995). Such spectra can also be obtained in SSQM by using cyclic shape-invariant potentials of period three (Sukhatme *et al.*, 1997). In this context, we recently showed that three appropriately chosen $\mathcal{A}^{(3)}(G(N))$ algebras provide a matrix realization of SSQM (Quesne and Vansteenkiste, 1999).

Another field wherein C_λ-extended oscillator algebras and their deformations may be of interest is the study of coherent (or squeezed) states in nonlinear quantum optics, wherein nonlinear oscillators are known to play an important role (McDermott and Solomon, 1994; Solomon, 1998; Man'ko *et al.*, 1997).

In the present paper, apart from studying some mathematical properties of C_λ-extended oscillator algebras, we deal with some important conceptual applications of these algebras. We indeed plan to show that they provide a bosonization (i.e., a realization in terms of only boson-like operators without fermion-like ones) of several variants of SSQM, namely parasupersymmetric quantum mechanics (PSSQM) of arbitrary order p (Rubakov and Spiridonov, 1988; Khare, 1992, 1993), pseudoSSQM (Beckers *et al.*, 1995a, b; Beckers and Debergh, 1995a, b), and orthosupersymmetric quantum mechanics (OSSQM) of order two (Khare *et al.*, 1993a). These results generalize that previously obtained for standard SSQM in terms of the Calogero–Vasiliev algebra (Brzeziński *et al.*, 1993; Plyushchay, 1996a, b).

In Section 2, we review the definition of C_λ-extended oscillator algebras, give their Casimir operators, and present some of their realizations. In Section 3, we classify their unitary irreducible representations (unirreps). In Sections 4–6, we consider their applications to PSSQM of arbitrary order p, pseudoSSQM, and OSSQM of order two, respectively. In Section 7, we construct some of their deformations. Finally, Section 8 contains the conclusion.

2. C_λ-EXTENDED OSCILLATOR ALGEBRAS

A C_λ-extended oscillator algebra $\mathcal{A}^{(\lambda)}$, where λ may take any value in the set $\{2, 3, 4, \ldots\}$, is defined (Quesne and Vansteenkiste, 1998) as an algebra generated by the operators I, $a^\dagger$, $a = (a^\dagger)^\dagger$, $N = N^\dagger$, and $T = (T^\dagger)^{-1}$, satisfying the relations

$$[N, a^\dagger] = a^\dagger, \qquad [N, T] = 0, \qquad T^\lambda = I \tag{2.1}$$

$$[a, a^\dagger] = I + \sum_{\mu=1}^{\lambda-1} \kappa_\mu T^\mu, \qquad a^\dagger T = e^{-i2\pi/\lambda} T a^\dagger \tag{2.2}$$

together with their Hermitian conjugates. Here κ_μ, $\mu = 1, 2, \ldots, \lambda - 1$, are some complex parameters restricted by the conditions $\kappa_\mu^* = \kappa_{\lambda-\mu}$ (so that there remain altogether $\lambda - 1$ independent real parameters), and T is the generator of the cyclic group of order λ, $C_\lambda = \{I, T, T^2, \ldots, T^{\lambda-1}\}$ (or, more precisely, the generator of a unitary representation thereof).

It is well known (Cornwell, 1984) that C_λ has λ inequivalent, one-dimensional matrix unirreps Γ^μ, $\mu = 0, 1, \ldots, \lambda - 1$, which are such that $\Gamma^\mu(T^\nu) = \exp(i2\pi\mu\nu/\lambda)$ for any $\nu = 0, 1, \ldots, \lambda - 1$. The projection operator on the carrier space of Γ^μ may be written as

$$P_\mu = \frac{1}{\lambda} \sum_{\nu=0}^{\lambda-1} (\Gamma^\mu(T^\nu))^* T^\nu = \frac{1}{\lambda} \sum_{\nu=0}^{\lambda-1} e^{-i2\pi\mu\nu/\lambda} T^\nu \tag{2.3}$$

and conversely T^ν, $\nu = 0, 1, \ldots, \lambda - 1$, may be expressed in terms of the P_μ as

$$T^\nu = \sum_{\mu=0}^{\lambda-1} e^{i2\pi\mu\nu/\lambda} P_\mu \tag{2.4}$$

The algebra defining relations (2.1) and (2.2) may therefore be rewritten in terms of I, $a^\dagger$, a, N, and $P_\mu = P_\mu{}^\dagger$, $\mu = 0, 1, \ldots, \lambda - 1$, as

$$[N, a^\dagger] = a^\dagger, \qquad [N, P_\mu] = 0, \qquad a^\dagger P_\mu = P_{\mu+1} a^\dagger \tag{2.5}$$

$$\sum_{\mu=0}^{\lambda-1} P_\mu = I, \qquad P_\mu P_\nu = \delta_{\mu,\nu} P_\mu \tag{2.6}$$

$$[a, a^\dagger] = I + \sum_{\mu=0}^{\lambda-1} \alpha_\mu P_\mu \tag{2.7}$$

where we use the convention $P_{\mu'} = P_\mu$ if $\mu' - \mu = 0 \bmod \lambda$ (and similarly for other operators or parameters labeled by μ, μ'). Equations (2.5)–(2.7) depend upon λ real parameters α_μ, $\mu = 0, 1, \ldots, \lambda - 1$, defined in terms of the κ_μ by

$$\alpha_\mu = \sum_{\nu=1}^{\lambda-1} \exp(i2\pi\mu\nu/\lambda)\kappa_\nu, \qquad \mu = 0, 1, \ldots, \lambda - 1 \tag{2.8}$$

and restricted by the condition

$$\sum_{\mu=0}^{\lambda-1} \alpha_\mu = 0 \tag{2.9}$$

Hence, we may eliminate one of them, e.g., $\alpha_{\lambda-1}$, and denote the algebra by $\mathcal{A}_{\alpha_0\alpha_1\ldots\alpha_{\lambda-2}}^{(\lambda)}$. It will, however, often prove convenient to work instead with the λ dependent parameters $\alpha_0, \alpha_1, \ldots, \alpha_{\lambda-1}$.

From Eqs. (2.1) and (2.2), or (2.5)–(2.7), it is easy to check that $\mathcal{A}_{\alpha_0\alpha_1\ldots\alpha_{\lambda-2}}^{(\lambda)}$ admits the following Casimir operators:

$$\mathcal{C}_1 = e^{i2\pi N} \tag{2.10}$$

$$\mathcal{C}_2 = e^{-i2\pi N/\lambda} T = \sum_{\mu=0}^{\lambda-1} e^{-i2\pi(N-\mu)/\lambda} P_\mu \tag{2.11}$$

$$\mathcal{C}_3 = N + \sum_{\mu=0}^{\lambda-1} \beta_\mu P_\mu - a^\dagger a \tag{2.12}$$

where

$$\beta_\mu = \sum_{\nu=0}^{\mu-1} \alpha_\nu, \qquad \mu = 1, 2, \ldots, \lambda - 1 \tag{2.13}$$

and $\beta_0 = \beta_\lambda = 0$. The first two operators are not functionally independent since

$$\mathscr{C}_1 \mathscr{C}_2^\lambda = I \tag{2.14}$$

From Eq. (2.11), it follows that the cyclic group generator T can be rewritten in terms of N and $\mathscr{C}_2$ as

$$T = e^{i2\pi N/\lambda} \mathscr{C}_2 \tag{2.15}$$

The simplest realization of the cyclic group C_λ uses functions of N. By taking $\mathscr{C}_2 = I$ in Eq. (2.15) and using Eq. (2.3), we obtain

$$T = e^{i2\pi N/\lambda}, \quad P_\mu = \frac{1}{\lambda} \sum_{\nu=0}^{\lambda-1} e^{i2\pi\nu(N-\mu)/\lambda}, \qquad \mu = 0, 1, \ldots, \lambda - 1 \tag{2.16}$$

With such a choice, $\mathscr{A}_{\alpha_0\alpha_1\ldots\alpha_{\lambda-2}}^{(\lambda)}$ becomes a GDOA $\mathscr{A}^{(\lambda)}(G(N))$, i.e., an algebra generated by I, $a^\dagger$, $a = (a^\dagger)^\dagger$, and $N = N^\dagger$, subject to the relations

$$[N, a^\dagger] = a^\dagger, \qquad [a, a^\dagger] = G(N) \tag{2.17}$$

where $G(N)$ is some Hermitian, analytic function of N (Quesne and Vansteenkiste, 1995). In the present case,

$$G(N) = I + \sum_{\mu=0}^{\lambda-1} \alpha_\mu P_\mu \tag{2.18}$$

where P_μ is given by Eq. (2.16).

According to the GDOA general theory (see Quesne and Vansteenkiste, 1995, 1996, 1997, and references quoted therein), one may define a structure function $F(N)$ which is the solution of the difference equation $F(N + 1) - F(N) = G(N)$ such that $F(0) = 0$. For $G(N)$ given in Eq. (2.18), one finds

$$F(N) = N + \sum_{\mu=0}^{\lambda-1} \beta_\mu P_\mu \tag{2.19}$$

where β_μ is defined in Eq. (2.13). From Eq. (2.19), it follows that the two Casimir operators $\mathscr{C}_1$, $\mathscr{C}_3$ of Eqs. (2.10), (2.12) reduce to the well-known Casimir operators $U = \exp(i2\pi N)$ and $\mathscr{C} = F(N) - a^\dagger a$, respectively (Quesne and Vansteenkiste, 1996, 1997).

It is worth noting that there exist other realizations of C_λ which may be interesting in some physical applications. We mention two of them.

The first one uses functions of spin-s operators, where $s = (\lambda - 1)/2$. Denoting as usual the spin operators [generating an $su(2)$ Lie algebra] by S_i, $i = 1, 2, 3$, it is obvious that the operators

$$P_\mu = \prod_{\substack{\sigma=-(\lambda-1)/2 \\ \sigma\neq(\lambda-2\mu-1)/2}}^{(\lambda-1)/2} \frac{S_3 - \sigma}{\frac{1}{2}(\lambda - 2\mu - 1) - \sigma}, \qquad \mu = 0, 1, \ldots, \lambda - 1 \tag{2.20}$$

acting in spin space, project on the spin components $\sigma = (\lambda - 1)/2$, $(\lambda - $

$3)/2, \ldots, (\lambda - 2\mu - 1)/2, \ldots, -(\lambda - 1)/2$, respectively. The corresponding realization of the C_λ generator T is obtained from Eq. (2.4) in the form

$$T = \sum_{\mu=0}^{\lambda-1} e^{i2\pi\mu/\lambda} \left(\prod_{\substack{\nu=0 \\ \nu\neq\mu}}^{\lambda-1} \frac{2S_3 - \lambda + 2\nu + 1}{2(\nu - \mu)} \right) \tag{2.21}$$

By using the $(2s + 1) \times (2s + 1)$ matrix representation of S_3, $S_3 = $ diag $(s, s - 1, \ldots, - s)$, we get another realization of C_λ in terms of $\lambda \times \lambda$ matrices,

$$T = \sum_{\mu=0}^{\lambda-1} e^{i2\pi\mu/\lambda} e_{\mu+1,\mu+1}, \qquad P_\mu = e_{\mu+1,\mu+1} \tag{2.22}$$

where e_{ij} denotes the $\lambda \times \lambda$ matrix with 1 in row i and column j, and zeros everywhere else.

Note that when considering such realizations of C_λ, the remaining $\mathcal{A}^{(\lambda)}_{\alpha_0\alpha_1\ldots\alpha_{\lambda-2}}$ generators would either act in both configuration and spin spaces or be $\lambda \times \lambda$ operator-valued matrices.

For $\lambda = 2$, the last relation in Eq. (2.1) and those in Eq. (2.2) become

$$T^2 = I, \qquad \{a^\dagger, T\} = 0, \qquad [a, a^\dagger] = I + \kappa_1 T = I + \alpha_0(P_0 - P_1) \tag{2.23}$$

where $P_0 = (I + T)/2$, $P_1 = (I - T)/2$, and $\kappa_1, \alpha_0 \in \mathbb{R}$. In the corresponding GDOA, the operator T is given by $T = \exp(i\pi N)$, which amounts to Klein operator $K = (-1)^N$, since as shown in the next section, the eigenvalues of N are integer in the $\mathcal{A}^{(2)}(G(N))$ unirreps. In the matrix realization (2.22), T is represented by the Pauli spin matrix σ_3, while $a^\dagger$, a can be expressed in terms of σ_1, σ_2, and some differential operators (Bagchi, 1994).

For $\lambda = 3$, the counterpart of Eq. (2.23) reads

$$T^3 = I, \qquad a^\dagger T = e^{-i2\pi/3} T a^\dagger \tag{2.24}$$

$$[a, a^\dagger] = I + \kappa_1 T + \kappa_1^* T^2 = I + \alpha_0 P_0 + \alpha_1 P_1 - (\alpha_0 + \alpha_1)P_2 \tag{2.25}$$

where $P_0 = (I + T + T^2)/3$, $P_1 = (I + e^{-i2\pi/3}T + e^{-i4\pi/3}T^2)/3$, $P_2 = (I + e^{-i4\pi/3}T + e^{-i2\pi/3}T^2)/3$, $\kappa_1 \in \mathbb{C}$, and $\alpha_0, \alpha_1 \in \mathbb{R}$. In the GDOA realization, the operator T is given by $T = \exp(i2\pi N/3)$, so that $G(N) = I + 2(\Re e \ \kappa_1) \cos(2\pi N/3) - 2(\Im m \ \kappa_1) \sin(2\pi N/3)$. In the matrix realization (2.22), T is represented by the matrix diag $(1, e^{i2\pi/3}, e^{i4\pi/3})$. Explicit expressions of $a^\dagger$, a are still unknown.

In the remainder of this paper, we shall concentrate on the abstract definition of $\mathcal{A}^{(\lambda)}_{\alpha_0\alpha_1\ldots\alpha_{\lambda-2}}$ or its GDOA realization $\mathcal{A}^{(\lambda)}(G(N))$.

3. UNIRREPS OF C_λ-EXTENDED OSCILLATOR ALGEBRAS

The purpose of the present section is to provide a classification of the $\mathcal{A}^{(\lambda)}_{\alpha_0 \alpha_1 \ldots \alpha_{\lambda-2}}$ unirreps. To carry out this program, it proves convenient to first consider the corresponding GDOA $\mathcal{A}^{(\lambda)}(G(N))$, defined in Eqs. (2.16)–(2.18).

3.1. Unirreps of $\mathcal{A}^{(\lambda)}(G(N))$

As a consequence of Eq. (2.14) and of the assumption $\mathcal{C}_2 = I$, the first Casimir operator $U = \mathcal{C}_1$ of $\mathcal{A}^{(\lambda)}(G(N))$ reduces to I; hence the eigenvalues of N are integer. As usual, we shall restrict ourselves to those unirreps wherein they are nondegenerate (Rideau, 1992; Quesne and Vansteenkiste, 1996, 1997).[4]

Let us start with a normalized simultaneous eigenvector $|c, n_0\rangle$ of the Casimir operator $\mathcal{C} = \mathcal{C}_3$, defined in Eq. (2.12), and of the number operator N, corresponding to the eigenvalues $c \in \mathbb{R}$ and $n_0 \in \mathbb{Z}$, respectively. From Eqs. (2.5)–(2.7), it results that as long as they are nonvanishing, the vectors

$$|c, n_0 + n\rangle = \begin{cases} (a^\dagger)^n |c, n_0\rangle, & \text{if } n = 0, 1, \ldots \\ a^{-n} |c, n_0\rangle, & \text{if } n = -1, -2, \ldots \end{cases} \quad (3.1)$$

satisfy the relations

$$\mathcal{C}|c, n_0 + n\rangle = c|c, n_0 + n\rangle, \qquad N|c, n_0 + n\rangle = (n_0 + n)|c, n_0 + n\rangle \quad (3.2)$$

$$a^\dagger a|c, n_0 + n\rangle = \lambda_n |c, n_0 + n\rangle, \qquad aa^\dagger|c, n_0 + n\rangle = \lambda_{n+1}|c, n_0 + n\rangle \quad (3.3)$$

where

$$\lambda_n = F(n_0 + n) - c \quad (3.4)$$

In any unirrep, only nonnegative values of λ_n are allowed. From Eq. (2.19), it is clear that the unirrep carrier space $\mathcal{S}$ is $\mathbb{Z}_\lambda$-graded: $\mathcal{S} = \Sigma_{\mu=0}^{\lambda-1} \oplus \mathcal{S}_\mu$, where $\mathcal{S}_\mu = \{|c, n_0 + n\rangle \,|\, n_0 + n = \mu \bmod \lambda\}$. Hence, we have to discuss the unitarity conditions $\lambda_n \geq 0$ separately in each $\mathcal{S}_\mu$ subspace. Since the structure function $F(N)$ is an increasing linear function of N in each S_μ, it is obvious that the algebra has no infinite-dimensional bounded from above (BFA) nor unbounded (UB) unirreps (Quesne and Vansteenkiste, 1996, 1997). It therefore only remains to successively consider the cases of infinite-dimensional bounded from below (BFB) unirreps and of finite-dimensional (FD) ones.

In the case of BFB unirreps, the eigenvalues of N are $n_0, n_0 + 1, n_0 + 2, \ldots$, and the unitarity conditions reduce to

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[4] In a recent study (Guichardet, 1998), the assumption that the spectrum of N is nondegenerate was lifted for the Arik–Coon GDOA (Arik and Coon, 1976; Kuryshkin, 1980), but it was shown that this condition is automatically fulfilled.

$$\lambda_0 = 0, \qquad \lambda_n > 0, \qquad \text{if} \quad n = 1, 2, \ldots, \lambda - 1 \tag{3.5}$$

The first condition in Eq. (3.5) fixes the Casimir operator eigenvalue,

$$c = n_0 + \beta_{\mu_0} \tag{3.6}$$

where $\mu_0 \in \{0, 1, \ldots, \lambda - 1\}$ is defined by

$$n_0 = \mu_0 \bmod \lambda \tag{3.7}$$

while the second condition yields some restrictions on the algebra parameters,

$$\bar{\beta}_\nu - \bar{\beta}_{\mu_0} + 1 > 0, \qquad \text{if} \quad \nu = 0, 1, \ldots, \mu_0 - 1 \tag{3.8}$$

$$\bar{\beta}_\nu - \bar{\beta}_{\mu_0} > 0, \qquad \text{if} \quad \nu = \mu_0 + 1, \mu_0 + 2, \ldots, \lambda - 1 \tag{3.9}$$

where

$$\bar{\beta}_\mu = \frac{\beta_\mu + \mu}{\lambda} \tag{3.10}$$

In terms of the α_μ, Eqs. (3.6), (3.8), and (3.9) can be rewritten as

$$c = n_0 + \sum_{\nu=0}^{\mu_0 - 1} \alpha_\nu \tag{3.11}$$

and

$$\alpha_\nu < \lambda - \mu_0 + \nu - \sum_{\rho=\nu+1}^{\mu_0 - 1} \alpha_\rho, \qquad \text{if} \quad \nu = 0, 1, \ldots, \mu_0 - 1 \tag{3.12}$$

$$\alpha_\nu > \mu_0 - \nu - 1 - \sum_{\rho=\mu_0}^{\nu-1} \alpha_\rho, \qquad \text{if} \quad \nu = \mu_0, \mu_0 + 1, \ldots, \lambda - 2 \tag{3.13}$$

respectively.

Whenever the unitarity conditions are satisfied, normalized basis states of $\mathcal{S}$ can be constructed from the vectors (3.1), and are given by

$$\left| c, n_0 + n \right\rangle = [\mathcal{N}_n(c, n_0)]^{-1/2} |c, n_0 + n), \qquad n = 0, 1, 2, \ldots \tag{3.14}$$

where the normalization coefficient is

$$\mathcal{N}_n(c, n_0) = \prod_{i=1}^{n} \lambda_i = \prod_{i=1}^{n} [F(n_0 + i) - c] \tag{3.15}$$

By writing n as $n = k\lambda + \mu$, where $\mu \in \{0, 1, \ldots, \lambda - 1\}$, and $k \in \mathbb{N}$, $\mathcal{N}_n(c, n_0)$ can be expressed in terms of gamma functions as

$$\mathcal{N}_{k\lambda+\mu}(c, n_0) = \lambda^{k\lambda+\mu}\left(\prod_{\nu=0}^{\mu_0+\mu} \Gamma(\bar{\beta}_\nu - \bar{\beta}_{\mu_0} + k + 1)\right)$$

$$\times \left(\prod_{\nu=\mu_0+\mu+1}^{\lambda-1} \Gamma(\bar{\beta}_\nu - \bar{\beta}_{\mu_0} + k)\right)$$

$$\times \left(\prod_{\nu=0}^{\mu_0} \Gamma(\bar{\beta}_\nu - \bar{\beta}_{\mu_0} + 1)\right)^{-1}\left(\prod_{\nu=\mu_0+1}^{\lambda-1} \Gamma(\bar{\beta}_\nu - \bar{\beta}_{\mu_0})\right)^{-1}$$

$$\text{if} \quad \mu = 0, 1, \ldots, \lambda - \mu_0 - 1$$

$$= \lambda^{k\lambda+\mu}\left(\prod_{\nu=0}^{\mu_0+\mu-\lambda} \Gamma(\bar{\beta}_\nu - \bar{\beta}_{\mu_0} + k + 2)\right)$$

$$\times \left(\prod_{\nu=\mu_0+\mu-\lambda+1}^{\lambda-1} \Gamma(\bar{\beta}_\nu - \bar{\beta}_{\mu_0} + k + 1)\right)$$

$$\times \left(\prod_{\nu=0}^{\mu_0} \Gamma(\bar{\beta}_\nu - \bar{\beta}_{\mu_0} + 1)\right)^{-1}\left(\prod_{\nu=\mu_0+1}^{\lambda-1} \Gamma(\bar{\beta}_\nu - \bar{\beta}_{\mu_0})\right)^{-1}$$

$$\text{if} \quad \mu = \lambda - \mu_0, \lambda - \mu_0 + 1, \ldots, \lambda - 1 \qquad (3.16)$$

In the case of FD unirreps, the eigenvalues of N are $n_0, n_0 + 1, \ldots,$ $n_0 + d - 1$, where the dimension d may only take values in the set $\{1, 2, \ldots, \lambda - 1\}$. The unitarity conditions are then given by

$$\lambda_0 = 0, \qquad \lambda_n > 0 \qquad \text{if} \quad n = 1, 2, \ldots, d - 1, \qquad \lambda_d = 0 \qquad (3.17)$$

Defining μ_0 and $\bar{\beta}_\mu$ as before by Eqs. (3.7) and (3.10), respectively, we obtain

$$c = n_0 + \beta_{\mu_0} \qquad (3.18)$$

$$\bar{\beta}_\nu - \bar{\beta}_{\mu_0} > 0 \qquad \text{if} \quad \nu = \mu_0 + 1, \mu_0 + 2, \ldots, \mu_0 + d - 1 \qquad (3.19)$$

$$\bar{\beta}_{\mu_0+d} - \bar{\beta}_{\mu_0} = 0 \qquad (3.20)$$

for $\mu_0 = 0, 1, \ldots, \lambda - d - 1$, and

$$c = n_0 + d + \beta_{\mu_0-\lambda+d} \qquad (3.21)$$

$$\bar{\beta}_\nu - \bar{\beta}_{\mu_0-\lambda+d} > 0 \qquad \text{if} \quad \nu = 0, 1, \ldots, \mu_0 - \lambda + d - 1 \qquad (3.22)$$

$$\bar{\beta}_\nu - \bar{\beta}_{\mu_0} > 0 \qquad \text{if} \quad \nu = \mu_0 + 1, \mu_0 + 2, \ldots, \lambda - 1 \qquad (3.23)$$

$$\bar{\beta}_{\mu_0-\lambda+d} - \bar{\beta}_{\mu_0} + 1 = 0 \qquad (3.24)$$

for $\mu_0 = \lambda - d, \lambda - d + 1, \ldots, \lambda - 1$. In terms of the algebra parameters α_μ, Eqs. (3.18)–(3.20), and Eqs. (3.21)–(3.24) become

$$c = n_0 + \sum_{\nu=0}^{\mu_0-1} \alpha_\nu \tag{3.25}$$

$$\alpha_\nu > \mu_0 - \nu - 1 - \sum_{\rho=\mu_0}^{\nu-1} \alpha_\rho$$

$$\text{if} \quad \nu = \mu_0, \mu_0 + 1, \ldots, \mu_0 + d - 2 \tag{3.26}$$

$$\alpha_{\mu_0+d-1} = -d - \sum_{\rho=\mu_0}^{\mu_0+d-2} \alpha_\rho \tag{3.27}$$

for $\mu_0 = 0, 1, \ldots, \lambda - d - 1$, and

$$c = n_0 + d + \sum_{\nu=0}^{\mu_0-\lambda+d-1} \alpha_\nu \tag{3.28}$$

$$\alpha_\nu < \lambda - \mu_0 + \nu - d - \sum_{\rho=\nu+1}^{\mu_0-\lambda+d-1} \alpha_\rho$$

$$\text{if} \quad \nu = 0, 1, \ldots, \mu_0 - \lambda + d - 1 \tag{3.29}$$

$$\alpha_\nu > \mu_0 - \nu - 1 - \sum_{\rho=\mu_0}^{\nu-1} \alpha_\rho \quad \text{if} \quad \nu = \mu_0, \mu_0 + 1, \ldots, \lambda - 2 \tag{3.30}$$

$$\alpha_{\mu_0-1} = d - \sum_{\rho=\mu_0-\lambda+d}^{\mu_0-2} \alpha_\rho \tag{3.31}$$

for $\mu_0 = \lambda - d, \lambda - d + 1, \ldots, \lambda - 1$, respectively.

Normalized basis states of the carrier space $\mathcal{S}$ of a d-dimensional unirrep are given by Eqs. (3.14) and (3.15), where n now runs over the range $n = 0, 1, \ldots, d - 1$. The corresponding normalization coefficient $\mathcal{N}_n(c, n_0)$ can be rewritten as

$$\mathcal{N}_n(c, n_0) = \lambda^n \prod_{\nu=\mu_0+1}^{\mu_0+n} (\overline{\beta}_\nu - \overline{\beta}_{\mu_0}) \tag{3.32}$$

for $\mu_0 = 0, 1, \ldots, \lambda - d - 1$, and

$$\mathcal{N}_n(c, n_0) = \lambda^n \prod_{\nu=\mu_0+1}^{\mu_0+n} (\overline{\beta}_\nu - \overline{\beta}_{\mu_0-\lambda+d} - 1)$$

$$\text{if} \quad n = 1, 2, \ldots, \lambda - \mu_0 - 1$$

Table I. Classification of $\mathcal{A}^{(2)}(G(N))$ Unirreps[a]

Type	n_0	c	Condition
BFB	$2k_0$	n_0	$\alpha_0 > -1$
BFB	$2k_0 + 1$	$n_0 + \alpha_0$	$\alpha_0 < 1$
FD ($d = 1$)	$2k_0$	n_0	$\alpha_0 = -1$
FD ($d = 1$)	$2k_0 + 1$	$n_0 + 1$	$\alpha_0 = 1$

[a] Here k_0 may take any integer value.

$$= \lambda^n \left(\prod_{\nu=0}^{\mu_0+n-\lambda} (\overline{\beta}_\nu - \overline{\beta}_{\mu_0-\lambda+d}) \right) \left(\prod_{\nu=\mu_0+1}^{\lambda-1} (\overline{\beta}_\nu - \overline{\beta}_{\mu_0-\lambda+d} - 1) \right)$$

$$\text{if} \quad n = \lambda - \mu_0, \lambda - \mu_0 + 1, \ldots, d - 1 \qquad (3.33)$$

for $\mu_0 = \lambda - d, \lambda - d + 1, \ldots, \lambda - 1$.

Tables I–III give the detailed unirrep classification for $\lambda = 2$, $\lambda = 3$, and $\lambda = 4$, respectively.

Of special interest in physical applications are the Fock-space unirreps, characterized by $c = n_0 = 0$. Since in this case $\mu_0 = 0$, such representations exist whenever the algebra parameters satisfy the conditions

$$\sum_{\rho=0}^{\nu} \alpha_\rho > -\nu - 1 \qquad \text{if} \quad \nu = 0, 1, \ldots, \lambda - 2 \qquad (3.34)$$

in the BFB case, and

Table II. Classification of $\mathcal{A}^{(3)}(G(N))$ Unirreps[a]

Type	n_0	c	Condition
BFB	$3k_0$	n_0	$\alpha_0 > -1, \alpha_1 > -2 - \alpha_0$
BFB	$3k_0 + 1$	$n_0 + \alpha_0$	$\alpha_0 < 2, \alpha_1 > -1$
BFB	$3k_0 + 2$	$n_0 + \alpha_0 + \alpha_1$	$\alpha_0 < 1 - \alpha_1, \alpha_1 < 2$
FD ($d = 1$)	$3k_0$	n_0	$\alpha_0 = -1$
FD ($d = 1$)	$3k_0 + 1$	$n_0 + \alpha_0$	$\alpha_1 = -1$
FD ($d = 1$)	$3k_0 + 2$	$n_0 + 1$	$\alpha_1 = 1 - \alpha_0$
FD ($d = 2$)	$3k_0$	n_0	$\alpha_0 > -1, \alpha_1 = -2 - \alpha_0$
FD ($d = 2$)	$3k_0 + 1$	$n_0 + 2$	$\alpha_0 = 2, \alpha_1 > -1$
FD ($d = 2$)	$3k_0 + 2$	$n_0 + \alpha_0 + 2$	$\alpha_0 < -1, \alpha_1 = 2$

[a] Here k_0 may take any integer value.

Table III. Classification of $\mathcal{A}^{(4)}(G(N))$ Unirreps[a]

Type	n_0	c	Condition
BFB	$4k_0$	n_0	$\alpha_0 > -1$, $\alpha_1 > -2 - \alpha_0$, $\alpha_2 > -3 - \alpha_0 - \alpha_1$
BFB	$4k_0 + 1$	$n_0 + \alpha_0$	$\alpha_0 < 3$, $\alpha_1 > -1$, $\alpha_2 > -2 - \alpha_1$
BFB	$4k_0 + 2$	$n_0 + \alpha_0 + \alpha_1$	$\alpha_0 < 2 - \alpha_1$, $\alpha_1 < 3$, $\alpha_2 > -1$
BFB	$4k_0 + 3$	$n_0 + \alpha_0 + \alpha_1 + \alpha_2$	$\alpha_0 < 1 - \alpha_1 - \alpha_2$, $\alpha_1 < 2 - \alpha_2$, $\alpha_2 < 3$
FD ($d = 1$)	$4k_0$	n_0	$\alpha_0 = -1$
FD ($d = 1$)	$4k_0 + 1$	$n_0 + \alpha_0$	$\alpha_1 = -1$
FD ($d = 1$)	$4k_0 + 2$	$n_0 + \alpha_0 + \alpha_1$	$\alpha_2 = -1$
FD ($d = 1$)	$4k_0 + 3$	$n_0 + 1$	$\alpha_2 = 1 - \alpha_0 - \alpha_1$
FD ($d = 2$)	$4k_0$	n_0	$\alpha_0 > -1$, $\alpha_1 = -2 - \alpha_0$
FD ($d = 2$)	$4k_0 + 1$	$n_0 + \alpha_0$	$\alpha_1 > -1$, $\alpha_2 = -2 - \alpha_1$
FD ($d = 2$)	$4k_0 + 2$	$n_0 + 2$	$\alpha_1 = 2 - \alpha_0$, $\alpha_2 > -1$
FD ($d = 2$)	$4k_0 + 3$	$n_0 + \alpha_0 + 2$	$\alpha_0 < -1$, $\alpha_2 = 2 - \alpha_1$
FD ($d = 3$)	$4k_0$	n_0	$\alpha_0 > -1$, $\alpha_1 > -2 - \alpha_0$, $\alpha_2 = -3 - \alpha_0 - \alpha_1$
FD ($d = 3$)	$4k_0 + 1$	$n_0 + 3$	$\alpha_0 = 3$, $\alpha_1 > -1$, $\alpha_2 > -2 - \alpha_1$
FD ($d = 3$)	$4k_0 + 2$	$n_0 + \alpha_0 + 3$	$\alpha_0 < -1$, $\alpha_1 = 3$, $\alpha_2 > -1$
FD ($d = 3$)	$4k_0 + 3$	$n_0 + \alpha_0 + \alpha_1 + 3$	$\alpha_0 < -2 - \alpha_1$, $\alpha_1 < -1$, $\alpha_2 = 3$

[a] Here k_0 may take any integer value.

$$\sum_{\rho=0}^{\nu} \alpha_\rho > - \nu - 1 \qquad \text{if} \quad \nu = 0, 1, \ldots, d - 2 \qquad (3.35)$$

$$\sum_{\rho=0}^{d-1} \alpha_\rho = - d \qquad (3.36)$$

in the FD one. The former are of bosonic type. Apart from the trivial one-dimensional unirrep, the latter are of fermionic or order-p-parafermionic type according to whether $d = 2$ or $d = p + 1 \geq 3$. Note that parafermionic-type unirreps only appear for $\lambda \geq 4$.

In the bosonic Fock-space representation, it may be interesting to consider a bosonic oscillator Hamiltonian (Quesne and Vansteenkiste, 1998), defined in appropriate units by

$$H_0 = \tfrac{1}{2}\{a, a^\dagger\} \qquad (3.37)$$

By using Eqs. (2.5)–(2.7), and (2.12), we can rewrite H_0 in the equivalent forms

$$H_0 = a^\dagger a + \frac{1}{2}\left(I + \sum_{\mu=0}^{\lambda-1} \alpha_\mu P_\mu\right) = N + \frac{1}{2} I + \sum_{\mu=0}^{\lambda-1} \gamma_\mu P_\mu \qquad (3.38)$$

where the parameters γ_μ are defined by

$$\gamma_\mu \equiv \tfrac{1}{2}(\beta_\mu + \beta_{\mu+1}) = \begin{cases} \tfrac{1}{2}\alpha_0, & \text{if} \quad \mu = 0 \\ \Sigma_{\nu=0}^{\mu-1} \alpha_\nu + \tfrac{1}{2}\alpha_\mu, & \text{if} \quad \mu = 1, 2, \ldots, \lambda - 1 \end{cases}$$

(3.39)

The latter satisfy the relation

$$\sum_{\mu=0}^{\lambda-1} (-1)^\mu \gamma_\mu = 0 \tag{3.40}$$

deriving from Eq. (2.9), as well as the inequalities

$$\gamma_\mu > -\tfrac{1}{2}(2\mu + 1) \qquad \text{if} \quad \mu = 0, 1, \ldots, \lambda - 2 \tag{3.41}$$

$$\gamma_{\lambda-1} > -\tfrac{1}{2}(\lambda - 1) \tag{3.42}$$

coming from conditions (3.34).

The states $|n\rangle = |k\lambda + \mu\rangle$, given by Eq. (3.14) where $c = n_0 = 0$ are the eigenstates of H_0, corresponding to the eigenvalues

$$E_{k\lambda+\mu} = k\lambda + \mu + \gamma_\mu + \tfrac{1}{2}, \qquad k = 0, 1, 2, \ldots, \quad \mu = 0, 1, \ldots, \lambda - 1 \tag{3.43}$$

In each $\mathscr{F}_\mu = \{|k\lambda + \mu\rangle|k = 0, 1, 2, \ldots\}$ subspace of the $\mathbb{Z}_\lambda$-graded Fock space $\mathscr{F} = \Sigma_{\mu=0}^{\lambda-1} \oplus \mathscr{F}_\mu$, the spectrum of H_0 is harmonic, but the λ infinite sets of equally spaced energy levels, corresponding to $\mu = 0, 1, \ldots, \lambda - 1$, may be shifted with respect to each other by some amounts depending upon the algebra parameters $\alpha_0, \alpha_1, \ldots, \alpha_{\lambda-2}$, through their linear combinations $\gamma_0, \gamma_1, \ldots, \gamma_{\lambda-1}$. As a result, one may get nondegenerate spectra, as well as spectra exhibiting some $(\nu + 1)$-fold degeneracies, where ν may take any value in the set $\{1, 2, \ldots, \lambda - 1\}$ (Quesne and Vansteenkiste, 1998, 1999).

3.2. Unirreps of $\mathscr{A}^{(\lambda)}_{\alpha_0\alpha_1\ldots\alpha_{\lambda-2}}$

Let us now turn to the general case of $\mathscr{A}^{(\lambda)}_{\alpha_0\alpha_1\ldots\alpha_{\lambda-2}}$, defined in Eqs. (2.1) and (2.2). Since we do not assume $\mathscr{C}_2 = I$, the eigenvalues of N are no longer restricted to integer values. It can be shown, however, that they are discrete. The proof proceeds as in Jordan *et al.* (1963) and Quesne and Vansteenkiste (1997), and can be summarized as follows. The Casimir operator $\mathscr{C}_1$, defined in Eq. (2.10), is unitary, so that in any given unirrep its eigenvalue can be written as $\exp(i2\pi\nu_0)$, where $\nu_0 \in \mathbb{R}$. On the other hand, the eigenvalues of $\mathscr{C}_1$ can be determined from those of the Hermitian operator N. The spectral mapping theorem leads to eigenvalues of $\mathscr{C}_1$ of the form $\exp(i2\pi x)$, where $x \in \mathbb{R}$ are the eigenvalues of N. The equivalence of the two expressions for the eigenvalues of $\mathscr{C}_1$ implies that $x = \nu_0 + n$, $n \in \mathbb{Z}$, in any given unirrep,

which completes the proof. As in Section 3.1, we shall restrict ourselves to those unirreps wherein the spectrum of N is not only discrete, but also nondegenerate.

As in Eq. (3.1), the carrier space of any $\mathcal{A}^{(\lambda)}_{\alpha_0\alpha_1...\alpha_{\lambda-2}}$ unirrep can be constructed by successive applications of $a^\dagger$ or a on a normalized simultaneous eigenvector $|c, \gamma, \nu_0\rangle$ of N and of the Casimir operators $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ defined in Eqs. (2.10)–(2.12),

$$N|c, \gamma, \nu_0\rangle = \nu_0|c, \gamma, \nu_0\rangle \tag{3.44}$$

$$\mathcal{C}_1|c, \gamma, \nu_0\rangle = e^{i2\pi r_0}|c, \gamma, \nu_0\rangle \tag{3.45}$$

$$\mathcal{C}_2|c, \gamma, \nu_0\rangle = e^{i2\pi(-r_0+\gamma)/\lambda}|c, \gamma, \nu_0\rangle \tag{3.46}$$

$$\mathcal{C}_3|c, \gamma, \nu_0\rangle = c|c, \gamma, \nu_0\rangle \tag{3.47}$$

Here $c, \nu_0 \in \mathbb{R}$, $\gamma \in \{0, 1, \ldots, \lambda - 1\}$, $r_0 \in [0, 1)$, is defined by

$$\nu_0 = n_0 + r_0, \qquad n_0 \in \mathbb{Z} \tag{3.48}$$

and the eigenvalue of $\mathcal{C}_2$ is determined from Eq. (2.14).

Let us now introduce some new operators and parameters, defined by

$$N' \equiv N - r_0 I, \qquad a'^\dagger \equiv a^\dagger, \qquad a' \equiv a, \qquad T' \equiv e^{-i2\pi\gamma/\lambda}T \tag{3.49}$$

$$\kappa'_\mu \equiv e^{i2\pi\mu\gamma/\lambda}\kappa_\mu = \kappa'^*_{\lambda-\mu} \tag{3.50}$$

from which we obtain

$$P'_\mu \equiv \frac{1}{\lambda}\sum_{\nu=0}^{\lambda-1} e^{-i2\pi\mu\nu/\lambda}T'^\nu = P_{\mu+\gamma} \tag{3.51}$$

$$\alpha'_\mu \equiv \sum_{\nu=1}^{\lambda-1} e^{i2\pi\mu\nu/\lambda}\kappa'_\nu = \alpha_{\mu+\gamma} = \alpha'^*_\mu \tag{3.52}$$

It is obvious that N', $a'^\dagger$, a', T' (or P'_μ) satisfy the defining relations (2.1) and (2.2) [or (2.5)–(2.7)] of $\mathcal{A}^{(\lambda)}_{\alpha_0\alpha'_1...\alpha'_{\lambda-2}}$, where the primed parameters α'_μ are given by Eq. (3.52). The corresponding Casimir operators $\mathcal{C}'_1$, $\mathcal{C}'_2$, $\mathcal{C}'_3$ are found to be expressible in terms of the old ones $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$,

$$\mathcal{C}'_1 \equiv e^{i2\pi N'} = e^{-i2\pi r_0}\mathcal{C}_1 \tag{3.53}$$

$$\mathcal{C}'_2 \equiv e^{-i2\pi N'/\lambda}T' = e^{i2\pi(r_0-\gamma)/\lambda}\mathcal{C}_2 \tag{3.54}$$

$$\mathcal{C}'_3 \equiv N' + \sum_{\mu=0}^{\lambda-1}\beta'_\mu P'_\mu - a'^\dagger a' = \mathcal{C}_3 - (r_0 + \beta_\gamma)I \tag{3.55}$$

where $\beta'_\mu \equiv \Sigma_{\nu=0}^{\mu-1}\alpha'_\nu = \beta_{\mu+\gamma} - \beta_\gamma$.

Hence, the simultaneous eigenvector $|c, \gamma, \nu_0\rangle$ of N, $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ is also a simultaneous eigenvector of N', $\mathcal{C}'_1$, $\mathcal{C}'_2$, $\mathcal{C}'_3$, satisfying the relations

$$N'|c, \gamma, \nu_0\rangle = n_0|c, \gamma, \nu_0\rangle \tag{3.56}$$

$$\mathcal{C}'_1|c, \gamma, \nu_0\rangle = \mathcal{C}'_2|c, \gamma, \nu_0\rangle = |c, \gamma, \nu_0\rangle \tag{3.57}$$

$$\mathcal{C}'_3|c, \gamma, \nu_0\rangle = c'|c, \gamma, \nu_0\rangle \tag{3.58}$$

where

$$c' = c - r_0 - \beta_\gamma \tag{3.59}$$

From Section 3.1, it follows that such a state may be identified with the starting eigenvector $|c', n_0\rangle$ of some unirrep of the GDOA $\mathcal{A}^{(\lambda)}(G'(N'))$, where $G'(N') = I + \Sigma_{\mu=0}^{\lambda-1} \alpha'_\mu P'_\mu$. Since $a'^\dagger = a^\dagger$ and $a' = a$, this correspondence between $|c, \gamma, \nu_0\rangle$ and $|c', n_0\rangle$ extends to the remaining basis states of the $\mathcal{A}^{(\lambda)}_{\alpha_0\alpha_1...\alpha_{\lambda-2}}$ and $\mathcal{A}^{(\lambda)}(G'(N'))$ unirreps built on such vectors, respectively.

We conclude that to every BFB (or FD) unirrep of $\mathcal{A}^{(\lambda)}(G'(N'))$, specified by some minimal N' eigenvalue $n_0 \in \mathbb{Z}$ (and some dimension d), we may associate an infinite number of BFB (or FD) unirreps of $\mathcal{A}^{(\lambda)}_{\alpha_0\alpha_1...\alpha_{\lambda-2}}$, characterized by minimal N eigenvalues $\nu_0 = n_0 + r_0$, $r_0 \in [0, 1)$, as well as $\mathcal{C}_2$ eigenvalues $\exp[i2\pi(-r_0 + \gamma)/\lambda]$, $\gamma \in \{0, 1, \ldots, \lambda - 1\}$ (and the same dimension d). The eigenvalues of the corresponding Casimir operators $\mathcal{C}'_3 = \mathcal{C}'$ and $\mathcal{C}_3$ are connected by Eq. (3.59). Furthermore, all the $\mathcal{A}^{(\lambda)}_{\alpha_0\alpha_1...\alpha_{\lambda-2}}$ unirreps are obtained by this mapping procedure.

4. APPLICATION OF C_λ-EXTENDED OSCILLATOR ALGEBRAS TO PSSQM OF ORDER $p = \lambda - 1$

PSSQM of order two was introduced by Rubakov and Spiridonov (1988) as a generalization of SSQM (Witten, 1981), obtained by combining standard bosons with parafermions of order two (Green, 1953; Ohnuki and Kamefuchi, 1982) instead of standard fermions. Its extension to arbitrary order p, due to Khare (1992, 1993), is described in terms of parasupercharge operators Q, $Q^\dagger$ and a parasupersymmetric Hamiltonian $\mathcal{H}$ satisfying the relations

$$Q^{p+1} = 0 \qquad (\text{with } Q^p \neq 0) \tag{4.1}$$

$$[\mathcal{H}, Q] = 0 \tag{4.2}$$

$$Q^p Q^\dagger + Q^{p-1} Q^\dagger Q + \cdots + Q Q^\dagger Q^{p-1} + Q^\dagger Q^p = 2p Q^{p-1} \mathcal{H} \tag{4.3}$$

and their Hermitian conjugates.

As shown by Bagchi $et\ al.$ (1997), PSSQM of order p can be reformulated in terms of p super (rather than parasuper) charges Q_ν, $\nu = 1, 2, \ldots, p$, all

of which satisfy $Q_\nu^2 = 0$ and commute with $\mathcal{H}$. However, unlike in usual SSQM, $\mathcal{H}$ cannot be simply expressed in terms of the p supercharges (except in a very special case to be reviewed below). More specifically, let us set

$$Q = \sum_{\nu=1}^{p} \sigma_\nu Q_\nu \qquad (4.4)$$

where σ_ν are some complex constants, and Q_ν, $\nu = 1, 2, \ldots, p$, are assumed to satisfy the relations

$$Q_\nu Q_{\nu'} = \delta_{\nu',\nu+1} Q_\nu Q_{\nu+1} \qquad (4.5)$$

$$Q_\nu Q_{\nu'}^\dagger = \delta_{\nu',\nu} Q_\nu Q_\nu^\dagger \qquad (4.6)$$

$$Q_\nu^\dagger Q_{\nu'} = \delta_{\nu',\nu} Q_\nu^\dagger Q_\nu \qquad (4.7)$$

Then, the operator Q defined in Eq. (4.4) satisfies Eqs. (4.1)–(4.3) if

$$\sigma_\nu \neq 0, \qquad \nu = 1, 2, \ldots, p \qquad (4.8)$$

$$[\mathcal{H}, Q_\nu] = 0, \qquad \nu = 1, 2, \ldots, p \qquad (4.9)$$

$$\left(\prod_{\nu=1}^{p-1} \sigma_\nu \right) Q_1 \Sigma + \left(\prod_{\nu=2}^{p} \sigma_\nu \right) \Sigma Q_p$$

$$= 2p \left[\left(\prod_{\nu=1}^{p-1} \sigma_\nu \right) Q_1 Q_2 \cdots Q_{p-1} + \left(\prod_{\nu=2}^{p} \sigma_\nu \right) Q_2 Q_3 \cdots Q_p \right] \mathcal{H} \qquad (4.10)$$

where

$$\Sigma \equiv |\sigma_1|^2 Q^\dagger Q_1 Q_2 \cdots Q_{p-1}$$

$$+ \sum_{\nu=2}^{p-1} |\sigma_\nu|^2 Q_2 Q_3 \cdots Q_\nu Q_\nu^\dagger Q_\nu Q_{\nu+1} \cdots Q_{p-1}$$

$$+ |\sigma_p|^2 Q_2 Q_3 \cdots Q_p Q_p^\dagger \qquad (4.11)$$

In the standard realization of PSSQM related to parafermions of order p (Khare, 1992, 1993), $\sigma_\nu = 1$, and Q_ν, $Q_\nu^\dagger$, $\mathcal{H}$ are represented by $(p + 1) \times (p + 1)$ matrices, whose elements are

$$(Q_\nu)_{\alpha,\beta} = (P - iW_\beta) \, \delta_{\alpha,\beta+1} \, \delta_{\beta,p+1-\nu} \qquad (4.12)$$

$$(Q_\nu^\dagger)_{\alpha,\beta} = (P + iW_\alpha) \, \delta_{\alpha,p+1-\nu} \, \delta_{\beta,\alpha+1} \qquad (4.13)$$

$$(\mathcal{H})_{\alpha,\beta} = \mathcal{H}_\alpha \, \delta_{\alpha,\beta} \qquad (4.14)$$

where $\alpha, \beta = 1, 2, \ldots, p + 1$. Here $P = -i\partial/\partial x$ is the momentum operator, $W_\nu(x)$, $\nu = 1, 2, \ldots, p$, are superpotentials, and

$$\mathcal{H}_\nu = \tfrac{1}{2}(P^2 + W_\nu^2 - W_\nu' + C_\nu), \qquad \nu = 1, 2, \ldots, p \quad (4.15)$$

$$\mathcal{H}_{p+1} = \tfrac{1}{2}(P^2 + W_p^2 + W_p' + C_p) \quad (4.16)$$

with $C_\nu \in \mathbb{R}$. The operator-valued matrices (4.12) and (4.13) automatically satisfy Eqs. (4.5)–(4.7), while Eqs. (4.9) and (4.10) impose the conditions

$$W_\nu^2 + W_\nu' + C_\nu = W_{\nu+1}^2 - W_{\nu+1}' + C_{\nu+1}, \qquad \nu = 1, 2, \ldots, p - 1$$

$$(4.17)$$

and

$$\sum_{\nu=1}^{p} C_\nu = 0 \quad (4.18)$$

respectively.

For arbitrary W_ν satisfying Eqs. (4.17) and (4.18), the spectrum of the parasupersymmetric Hamiltonian $\mathcal{H}$ is $(p + 1)$-fold degenerate at least starting from the pth excited state onward. The nature of the ground and the first $(p - 1)$ excited states depends, however, on the specific form of the W_ν. For the special choice $W_1 = W_2 = \cdots = W_p = \omega x$, $\mathcal{H}$ becomes the parasupersymmetric oscillator Hamiltonian, which can be realized in terms of bosons and parafermions of order p. Its ground state is nondegenerate, and has a negative energy, while the νth excited state for $\nu = 1, 2, \ldots, p - 1$ is $(\nu + 1)$-fold degenerate.

We now plan to show that the PSSQM algebra (4.1)–(4.3) can be realized in terms of the generators of $\mathcal{A}^{(\lambda)}(G(N))$, $\lambda = p + 1$, in their bosonic Fock-space representation [then the parameters $\alpha_0, \alpha_1, \ldots, \alpha_{\lambda-2}$ satisfy Eq. (3.34)]. This will prove that PSSQM of arbitrary order p can be bosonized, as is the case for standard SSQM (Brzeziński *et al.*, 1993; Plyushchay, 1996a, b; Beckers *et al.*, 1997) and PSSQM of order two (Quesne and Vansteenkiste, 1998).

In view of the results previously obtained for $p = 2$ (Quesne and Vansteenkiste, 1998), let us take as ansätze for the operators Q and $\mathcal{H}$ the expressions

$$Q = \sum_{\nu=1}^{p} \eta_{\mu+\nu} a^\dagger P_{\mu+\nu} \quad (4.19)$$

$$\mathcal{H} = H_0 + \frac{1}{2} \sum_{\nu=0}^{p} r_\nu P_\nu \quad (4.20)$$

where H_0 is the bosonic oscillator Hamiltonian (3.37) associated with the algebra $\mathcal{A}^{(p+1)}(G(N))$, $\eta_{\mu+\nu}$, $\nu = 1, 2, \ldots, p$, are some complex constants, and r_ν, $\nu = 0, 1, \ldots, p$, some real ones. The purpose of the last term on the right-hand side of Eq. (4.20) is to make the $p + 1$ families of H_0 equally spaced eigenvalues coincide at least starting from the pth excited state onward. Note that in Eqs. (4.19) and (4.20), μ takes some fixed, arbitrary value in the set $\{0, 1, \ldots, p\}$. The operators Q, $Q^\dagger$, $\mathcal{H}$ and all the quantities to be considered hereafter depend on this μ value, although for simplicity's sake we chose not to exhibit such a dependence explicitly by appending a μ index to them.

It is straightforward to see that the operators

$$Q_\nu = a^\dagger P_{p+1+\mu-\nu}, \qquad \nu = 1, 2, \ldots, p \qquad (4.21)$$

satisfy Eqs. (4.5)–(4.7); hence Q, as defined by Eq. (4.19), can be written in the form (4.4) by setting

$$\sigma_\nu = \eta_{p+1+\mu-\nu}, \qquad \nu = 1, 2, \ldots, p \qquad (4.22)$$

Equation (4.8) leads to the restriction

$$\eta_{\mu+\nu} \neq 0, \qquad \nu = 1, 2, \ldots, p \qquad (4.23)$$

After some calculations, one finds that Eqs. (4.9) and (4.10) are equivalent to the conditions

$$r_{\mu+\nu} = 2 + \alpha_{\mu+\nu} + \alpha_{\mu+\nu+1} + r_{\mu+\nu+1}, \qquad \nu = 1, 2, \ldots, p \qquad (4.24)$$

and

$$\sum_{\nu=1}^{p} \left|\eta_{\mu+\nu}\right|^2 = 2p \qquad (4.25)$$

$$\sum_{\nu=2}^{p} \left|\eta_{\mu+\nu}\right|^2 \left(\nu - 1 + \sum_{\rho=0}^{\nu-2} \alpha_{\mu+\rho+2}\right) = p(1 + \alpha_{\mu+2} + r_{\mu+2}) \qquad (4.26)$$

respectively.

Equation (4.24) is a nonhomogeneous system of p linear equations in $(p + 1)$ unknowns $r_{\mu+\nu}$, $\nu = 0, 1, \ldots, p$. Its solution yields p of them in terms of the remaining one, e.g., r_μ, $r_{\mu+1}$, $r_{\mu+3}$, $\ldots$, $r_{\mu+p}$ in terms of $r_{\mu+2}$:

$$r_\mu = -2(p - 1) - \alpha_\mu - \alpha_{\mu+2} - 2 \sum_{\rho=3}^{p} \alpha_{\mu+\rho} + r_{\mu+2}$$

$$= -2(p - 1) - 2\gamma_\mu + 2\gamma_{\mu+2} + r_{\mu+2} \qquad (4.27)$$

$$r_{\mu+1} = 2 + \alpha_{\mu+1} + \alpha_{\mu+2} + r_{\mu+2} = 2 - 2\gamma_{\mu+1} + 2\gamma_{\mu+2} + r_{\mu+2} \qquad (4.28)$$

$$r_{\mu+\nu} = -2(\nu - 2) - \alpha_{\mu+2} - 2\sum_{\rho=3}^{\nu-1} \alpha_{\mu+\rho} - \alpha_{\mu+\nu} + r_{\mu+2}$$

$$= -2(\nu - 2) + 2\gamma_{\mu+2} - 2\gamma_{\mu+\nu} + r_{\mu+2}, \qquad \nu = 3, 4, \ldots, p \quad (4.29)$$

where γ_μ is defined in Eq. (3.39).

Equation (4.25) restricts the range of $|\eta_{\mu+\nu}|^2$, $\nu = 1, 2, \ldots, p$, while Eq. (4.26) fixes the value of $r_{\mu+2}$ in terms of the latter and the algebra parameters. We conclude that it is possible to find values of $\eta_{\mu+\nu}$ and r_ν in Eqs. (4.19) and (4.20) so that Eqs. (4.1)–(4.3) are satisfied. Choosing, for instance,

$$|\eta_{\mu+\nu}|^2 = 2, \qquad \nu = 1, 2, \ldots, p \quad (4.30)$$

we obtain

$$r_{\mu+2} = \frac{1}{p}\left[(p - 2)\alpha_{\mu+2} + 2\sum_{\nu=3}^{p} (p - \nu + 1)\alpha_{\mu+\nu} + p(p - 2) \right] \quad (4.31)$$

or

$$r_{\mu+2} = \frac{1}{p}\left\{ 2[1 - (-1)^p]\sum_{\nu=0}^{\mu+1} (-1)^{\mu+1-\nu}\gamma_\nu - 2[p - 1 - (-1)^p]\gamma_{\mu+2} \right.$$

$$\left. + 2\sum_{\nu=3}^{p-2} [1 + (-1)^{p-\nu}]\gamma_{\mu+\nu} + 4\gamma_{\mu+p} + p(p - 2) \right\} \quad (4.32)$$

In going from Eq. (4.31) to Eq. (4.32), we used the inverse of Eq. (3.39), namely

$$\alpha_\mu = \begin{cases} 2\gamma_0, & \text{if} \quad \mu = 0 \\ 4\sum_{\nu=0}^{\mu-1}(-1)^{\mu-\nu}\gamma_\nu + 2\gamma_\mu, & \text{if} \quad \mu = 1, 2, \ldots, \lambda - 1 \end{cases} \quad (4.33)$$

From Eqs. (3.38) and (4.27)–(4.29) it follows that the parasupersymmetric Hamiltonian (4.20) can be rewritten as

$$\mathcal{H} = N + \frac{1}{2}(2\gamma_{\mu+2} + r_{\mu+2} - 2p + 3)I$$

$$+ \sum_{\nu=1}^{p} (p + 1 - \nu)P_{\mu+\nu} \quad (4.34)$$

where $r_{\mu+2}$ is given by Eq. (4.32). The eigenstates $|n\rangle = |k(p + 1) + \nu\rangle$, n, $k = 0, 1, 2, \ldots, \nu = 0, 1, \ldots, p$, of H_0 are also eigenstates of $\mathcal{H}$, corresponding to the eigenvalues

$$\mathcal{E}_{k(p+1)+\nu} = k(p+1) + \tfrac{1}{2}(2\gamma_{\mu+2} + r_{\mu+2} + 2\mu - 2p + 3)$$

$$\text{if} \quad \nu = 0, 1, \ldots, \mu \tag{4.35}$$

$$\mathcal{E}_{k(p+1)+\nu} = (k+1)(p+1) + \tfrac{1}{2}(2\gamma_{\mu+2} + r_{\mu+2} + 2\mu - 2p + 3)$$

$$\text{if} \quad \nu = \mu + 1, \mu + 2, \ldots, p \tag{4.36}$$

All the levels are therefore equally spaced. The ground state, corresponding to the energy

$$\mathcal{E}_0 = \mathcal{E}_1 = \cdots = \mathcal{E}_\mu = \tfrac{1}{2}(2\gamma_{\mu+2} + r_{\mu+2} + 2\mu - 2p + 3) \tag{4.37}$$

is $(\mu + 1)$-fold degenerate, whereas the excited states are $(p + 1)$-fold degenerate. Note that since μ may take any value in the set $\{0, 1, \ldots, p\}$, the ground-state degeneracy may accordingly vary between 1 and $p + 1$. Unbroken (resp. broken) PSSQM corresponds to $\mu = 0$ (resp. $\mu = 1, 2, \ldots, \text{or } p$).

To study the sign of the ground-state energy, we have to insert Eq. (4.32) into Eq. (4.37). The result reads

$$\mathcal{E}_0 = \mathcal{E}_1 = \cdots = \mathcal{E}_\mu$$

$$= \frac{1}{2p}\left[4 \sum_{\nu=0}^{(\mu-2)/2} \gamma_{2\nu+1} + 4 \sum_{\nu=(\mu+2)/2}^{[p/2]} \gamma_{2\nu} + p(2\mu - p + 1) \right]$$

$$\text{if} \quad \mu = 0, 2, \ldots, 2[p/2] \tag{4.38}$$

$$\mathcal{E}_0 = \mathcal{E}_1 = \cdots = \mathcal{E}_\mu$$

$$= \frac{1}{2p}\left[4 \sum_{\nu=0}^{(\mu-1)/2} \gamma_{2\nu} + 4 \sum_{\nu=(\mu+1)/2}^{[(p-1)/2]} \gamma_{2\nu+1} + p(2\mu - p + 1) \right]$$

$$\text{if} \quad \mu = 1, 3, \ldots, 2[(p-1)/2] + 1 \tag{4.39}$$

where $[a]$ denotes the largest integer contained in a, and $\sum_{\nu=a}^{b} \equiv 0$ if $a > b$. From the conditions (3.41) and (3.42) for the existence of the bosonic Fock-space representation, it follows that

$$\mathcal{E}_0 = \mathcal{E}_1 = \cdots = \mathcal{E}_\mu > \frac{1}{p}(p+1)(\mu - p + 1)$$

$$\text{if} \quad \mu = 0, 1, \ldots, p - 2 \tag{4.40}$$

$$\mathcal{E}_0 = \mathcal{E}_1 = \cdots = \mathcal{E}_\mu > 0 \quad \text{if} \quad \mu = p - 1, p \tag{4.41}$$

Since the right-hand side of Eq. (4.40) is negative, for $\mu = 0, 1, \ldots, p - 2$, the ground-state energy may be positive, null, or negative according to

the values taken by the algebra parameters. We therefore recover a well-known property of PSSQM of order $p \geq 2$: unlike in SSQM (corresponding to $p = 1$), the energy eigenvalues are not necessarily nonnegative, and there is no connection between the nonvanishing (resp. vanishing) ground-state energy and the broken (resp. unbroken) PSSQM.

As noted by Khare *et al.* (1993b), there is, however, a special case in the standard PSSQM realization (4.12)–(4.16) wherein this unsatisfactory situation does not occur, and moreover the parasupersymmetric Hamiltonian $\mathcal{H}$ can be expressed directly in terms of the parasupercharge operators Q, $Q^{\dagger}$, in contrast with Eq. (4.3). Whenever all the constants C_{ν} vanish in Eq. (4.17), one can indeed write $\mathcal{H}$ as

$$\mathcal{H} = \tfrac{1}{2}[(Q^{\dagger}Q - QQ^{\dagger})^2 + Q^{\dagger}Q^2Q^{\dagger}]^{1/2} \tag{4.42}$$

whose eigenvalues are necessarily nonnegative. Furthermore, its ground-state energy vanishes (resp. is positive) for unbroken (resp. broken) PSSQM.

Such a special case does have a counterpart in the present bosonic realization. By introducing Eqs. (4.19) and (4.20) into Eq. (4.42), and taking Eq. (4.30) into account, it is easy to show that Eq. (4.42) is equivalent to the following additional conditions:

$$r_{\mu} = -1 - \alpha_{\mu}, \qquad r_{\mu+1} = 1 + \alpha_{\mu+1},$$
$$r_{\mu+\nu} = 0, \qquad \nu = 2, 3, \ldots, p \tag{4.43}$$
$$\alpha_{\mu+\nu} = -1, \qquad \nu = 2, 3, \ldots, p \tag{4.44}$$

which can be checked to be compatible with the previous ones given in Eqs. (4.27)–(4.29), and (4.31).

However, the conditions (3.34) for the existence of the bosonic Fock-space representation are compatible with Eq. (4.44) only for $\mu = 0$ and $\mu = p$. In the former case, $\alpha_1 = p - 1 - \alpha_0$, $\alpha_2 = \alpha_3 = \cdots = \alpha_p = -1$, where $\alpha_0 > -1$, and from Eqs. (4.34) and (4.43),

$$\mathcal{H} = N + \sum_{\nu=1}^{p} (p + 1 - \nu)P_{\nu} \tag{4.45}$$

PSSQM is then unbroken, and the ground-state energy vanishes [$\mathcal{E}_0 = 0$ in accordance with Eq. (4.37), since $\gamma_2 = p - \tfrac{3}{2}$]. In the latter case, $\alpha_1 = \alpha_2 = \cdots = \alpha_{p-1} = -1$, $\alpha_p = p - 1 - \alpha_0$, where $\alpha_0 > -1$, and

$$\mathcal{H} = N + \sum_{\nu=0}^{p} (\alpha_0 + 1 - \nu)P_{\nu} \tag{4.46}$$

PSSQM is then broken, and the ground-state energy $\mathcal{E}_0 = \mathcal{E}_1 = \cdots \mathcal{E}_p =$

$\alpha_0 + 1$ [in accordance with Eq. (4.37), since $\gamma_{p+2} = \gamma_1 = \alpha_0 - \frac{1}{2}$] is positive, the ground state being $(p + 1)$-fold degenerate as all the excited states.

Furthermore, by using conditions (4.43) and (4.44), it can be shown that $\mathcal{H}$ can be rewritten in terms of the supercharges (4.21) as

$$\mathcal{H} = Q_1 Q_1^\dagger + \sum_{\nu=1}^{p} Q_\nu^\dagger Q_\nu \tag{4.47}$$

This result also has its counterpart in the standard PSSQM realization (Bagchi *et al.*, 1997).

Going back now to the general case corresponding to conditions (4.27)–(4.31) only, we note that Eq. (4.30), yielding the coefficients in the expansion of the parasupercharges (4.19), has many solutions. This is not surprising since Khare did show that in the standard PSSQM realization (4.12)–(4.16), $\mathcal{H}$ has in fact p (and not only one) conserved parasupercharges, as well as p bosonic constants (Khare, 1992, 1993). In other words, there exist p independent operators Q_r, $r = 1, 2, \ldots, p$, satisfying with $\mathcal{H}$ the set of equations (4.1)–(4.3), and p other independent operators I_t, $t = 2, 3, \ldots, p + 1$, commuting with $\mathcal{H}$, as well as among themselves. The former are obtained from Eqs. (4.4) and (4.12) by setting $\sigma_\nu = 1$ for $r = 1$ and $\sigma_\nu = 1 - 2\delta_{\nu,p+1-r}$ for $r = 2, 3, \ldots, p$, while the latter are given by $(I_t)_{\alpha,\beta} = \delta_{\alpha,\beta}(1 - 2\delta_{\alpha,t})$, where $t = 2, 3, \ldots, p + 1$, and $\alpha, \beta = 1, 2, \ldots, p + 1$. In addition, for any $r_k, r_{k+1}, \ldots, r_{k+p} \in \{1, 2, \ldots, p\}$,

$$Q_{r_k} Q_{r_{k+1}} \cdots Q_{r_{k+p}} = 0 \tag{4.48}$$

and for any $r \in \{1, 2, \ldots, p\}$, $t \in \{2, 3, \ldots, p + 1\}$,

$$[I_t, Q_r] = \sum_{s=1}^{p} d_{tr}^s Q_s \tag{4.49}$$

where d_{tr}^s are some real constants, e.g.,

$$d_{21}^1 = d_{22}^2 = 0, \qquad d_{21}^2 = d_{22}^1 = -2,$$
$$d_{31}^1 = -d_{31}^2 = -d_{32}^1 = d_{32}^2 = -1 \tag{4.50}$$

for $p = 2$. Finally, the Q_r satisfy some mixed multilinear relations generalizing Eq. (4.3), and involving $\mathcal{H}$ and the bosonic constants I_t. For $p = 2$, for instance, there are six such independent relations

$$I_3 Q_s^2 Q_r^\dagger + Q_s Q_r^\dagger Q_s + I_2 Q_r^\dagger Q_s^2 = 4 Q_r \mathcal{H} \tag{4.51}$$

$$Q_r Q_s Q_s^\dagger + Q_s Q_s^\dagger Q_r + I_2 Q_s^\dagger Q_r Q_s = 4 Q_r \mathcal{H} \tag{4.52}$$

$$I_3 Q_s Q_r Q_s^\dagger + Q_r Q_s^\dagger Q_s + Q_s^\dagger Q_s Q_r = 4 Q_r \mathcal{H} \tag{4.53}$$

where $(r, s) = (1, 2), (2, 1)$.

It is straightforward to show that the operators Q_r and I_t have also their counterpart in the present bosonic realization. Let us indeed consider the operators

$$Q_r = \sqrt{2} \sum_{\nu=1}^{p} b_r^\nu a^\dagger P_{\mu+\nu}, \qquad r = 1, 2, \ldots, p \tag{4.54}$$

$$I_t = \sum_{\nu=1}^{p+1} b_t^\nu P_{\mu+\nu}, \qquad t = 1, 2, \ldots, p + 1 \tag{4.55}$$

where

$$b_t^\nu = 1 - 2\delta_{t,\nu}(1 - \delta_{t,1}), \qquad t, \nu = 1, 2, \ldots, p + 1 \tag{4.56}$$

The b_t^ν taking values only in the set $\{-1, +1\}$, it is clear that each Q_r in Eq. (4.54) satisfies the PSSQM algebra (4.1)–(4.3) with Hamiltonian (4.34). It is also obvious that $I_2, I_3, \ldots, I_{p+1}$, as defined by Eq. (4.55), commute with the same, as well as among themselves, while I_1 reduces to the unit operator. Equation (4.48) directly follows for $n = p$ from the relation

$$Q_{r_k} Q_{r_{k+1}} \cdots Q_{r_{k+n}} = 2^{(n+1)/2}(a^\dagger)^{n+1} \sum_{\nu=1}^{p-n} B_\nu(r_k, r_{k+1}, \ldots, r_{k+n}) P_{\mu+\nu} \tag{4.57}$$

$$B_\nu(r_k, r_{k+1}, \ldots, r_{k+n}) \equiv \prod_{l=0}^{n} b_{r_{k+l}}^{\nu+n-l} \tag{4.58}$$

which can be proved by induction over n.

Considering now Eq. (4.49), we obtain from Eqs. (4.54) and (4.55)

$$[I_t, Q_r] = \sqrt{2} a^\dagger \sum_{\nu=1}^{p} c_{tr}^\nu P_{\mu+\nu}, \qquad c_{tr}^\nu \equiv (b_t^{\nu+1} - b_t^\nu) b_r^\nu \tag{4.59}$$

By combining this result with the inverse of Eq. (4.54),

$$\sqrt{2} a^\dagger P_{\mu+\nu} = \sum_{r=1}^{p} b_\nu^r Q_r \tag{4.60}$$

$$b_\nu^r \equiv \tfrac{1}{2}\{\delta_{\nu,1}[1 + (2 - p)\delta_{r,1}] + (1 - \delta_{\nu,1})(\delta_{r,1} - \delta_{\nu,r})\} \tag{4.61}$$

we get Eq. (4.49) with d_{tr}^s given by

$$d_{tr}^s = \sum_{\nu=1}^{p} c_{tr}^\nu b_\nu^s \tag{4.62}$$

For the special cases $p = 2$ and $p = 3$ considered by Khare (1992, 1993), this general formula yields the correct results [see, e.g., Eq. (4.50)].

Finally, for the mixed multilinear relations satisfied by the Q_r and I_t, let us consider a general relation of the type

$$I_{t_1}Q_{r_1}Q_{r_2}\ldots Q_{r_p}Q_s^\dagger + I_{t_2}Q_{r_2}Q_{r_3}\ldots Q_{r_p}Q_s^\dagger Q_{r_1}$$

$$+ \cdots + I_{t_p}Q_{r_p}Q_s^\dagger Q_{r_1}Q_{r_2}\ldots Q_{r_{p-1}}$$

$$+ I_{t_{p+1}}Q_s^\dagger Q_{r_1}Q_{r_2}\ldots Q_{r_p} = 2pQ_r^{p-1}\mathcal{H} \tag{4.63}$$

where $r_1, r_2, \ldots, r_p \in \{1, 2, \ldots, p\}$ and $t_1, t_2, \ldots, t_{p+1} \in \{1, 2, \ldots, p + 1\}$. It is clear that such a relation cannot be valid for any choice of the indices in the ranges indicated. To find to which choices it applies when definitions (4.54) and (4.55) are used, let us work out the conditions implied by Eq. (4.63).

After some calculations, one gets

$$\sum_{\nu=1}^{p} D_k^\nu = pB_k([r]^{p-1}), \qquad k = 1, 2 \tag{4.64}$$

$$\sum_{\nu=2}^{p} D_k^\nu\left(\nu - 1 + \sum_{\rho=0}^{\nu-2} \alpha_{\mu+\rho+2}\right) = \frac{p}{2} B_k([r]^{p-1})(1 + \alpha_{\mu+2} + r_{\mu+2}),$$

$$k = 1, 2 \tag{4.65}$$

where $[r]^{p-1}$ means that r is repeated $(p - 1)$ times, and

$$D_k^\nu \equiv b_{t_{\nu+2-k}}^{p+k-1} B_\nu(r_{\nu+2-k}, r_{\nu+3-k}, \ldots, r_p)b_s^\nu B_k(r_1, r_2, \ldots, r_{\nu+1-k}) \tag{4.66}$$

Since $B_k([r]^{p-1})$ and D_k^ν take values in the set $\{+1, -1\}$, Eq. (4.64) is satisfied if and only if

$$D_k^\nu = B_k([r]^{p-1}), \qquad k = 1, 2, \quad \nu = 1, 2, \ldots, p \tag{4.67}$$

Then Eq. (4.65) reduces to Eq. (4.26), where the choice (4.30) has been made; hence it is automatically fulfilled. We are therefore left with condition (4.67), where we note that

$$B_1([r]^{p-1}) = 2(\delta_{r,1} + \delta_{r,p}) - 1, \qquad B_2([r]^{p-1}) = 2\delta_{r,1} - 1 \tag{4.68}$$

We conclude that finding all mixed multilinear relations of type (4.63) amounts to determining all sets of b_t^ν coefficients satisfying Eqs. (4.66)–(4.68).

Once this has been done, it still remains to eliminate some dependent relations by taking into account identities such as

$$I_{r+1}Q_r = Q_1, \qquad r = 2, 3, \ldots, p \tag{4.69}$$

$$I_t Q_1 = Q_{t-1}, \qquad t = 3, 4, \ldots, p + 1 \tag{4.70}$$

$$Q_{r_1}Q_{r_2}\ldots Q_{r_p}Q_s^\dagger = I_t Q_{r_1}Q_{r_2}\ldots Q_{r_p}Q_s^\dagger,$$

$$t = 1, 2, \ldots, p \tag{4.71}$$

$$Q_{r_k}Q_{r_{k+1}} \cdots Q_{r_p}Q_s^\dagger Q_{r_1}Q_{r_2} \cdots Q_{r_{k-1}} = I_t Q_{r_k}Q_{r_{k+1}} \cdots Q_{r_p}Q_s^\dagger Q_{r_1}Q_{r_2} \cdots Q_{r_{k-1}}$$

$$k = 2, 3, \ldots, p,$$

$$t = 1, 2, \ldots, p - 1 \tag{4.72}$$

$$Q_s^\dagger Q_{r_1}Q_{r_2} \cdots Q_{r_p} = I_t Q_s^\dagger Q_{r_1}Q_{r_2} \cdots Q_{r_p},$$

$$t = 1, 2, \ldots, p - 1, p + 1 \tag{4.73}$$

By proceeding in this way for $p = 2$, one gets the six relations given in Eqs. (4.51)–(4.53). The $p = 3$ case can be dealt with in a similar way, giving back the results of Khare (1993).

As a final point, let us note that there exists an alternative approach to PSSQM of order p, due to Beckers and Debergh (1990), wherein Eq. (4.3) is replaced by the cubic equation

$$[Q, [Q^\dagger, Q]] = 2Q\mathcal{H} \tag{4.74}$$

while Eqs. (4.1) and (4.2) remain the same. We proved elsewhere (Quesne and Vansteenkiste, 1998) that in the $p = 2$ case, Beckers–Debergh PSSQM algebra can only be realized by those $\mathcal{A}^{(3)}(G(N))$ algebras that simultaneously bosonize Rubakov–Spiridonov–Khare PSSQM algebra. For this reason, we do not consider here that alternative approach to PSSQM of order p.

5. APPLICATION OF C_3-EXTENDED OSCILLATOR ALGEBRAS TO PSEUDOSSQM

PseudoSSQM was introduced by Beckers *et al.* (1995a, b) (see also Beckers and Debergh, 1995a, b) in a study of relativistic vector mesons interacting with an external constant magnetic field, wherein the reality of energy eigenvalues was required. In the nonrelativistic limit, their theory leads to a pseudosupersymmetric oscillator Hamiltonian which can be realized in terms of bosons and pseudofermions, where the latter are intermediate between standard fermions and parafermions of order two. It is then possible to formulate a pseudoSSQM characterized by a pseudosupersymmetric Hamiltonian $\mathcal{H}$ and pseudosupercharge operators Q, $Q^\dagger$, satisfying the relations

$$Q^2 = 0 \tag{5.1}$$

$$[\mathcal{H}, Q] = 0 \tag{5.2}$$

$$QQ^\dagger Q = 4c^2 Q\mathcal{H} \tag{5.3}$$

and their Hermitian conjugates, where c is some real constant. The first two

relations in Eqs. (5.1), (5.2) are the same as those occurring in SSQM, whereas the third one in Eq. (5.3) is similar to the multilinear relation valid in PSSQM of order two. Actually, for $c = 1$ or $1/2$, it is compatible with Eq. (4.3) or (4.74), respectively.

We will now show that the pseudoSSQM algebra (5.1)–(5.3) can be realized in terms of the generators of $\mathcal{A}^{(3)}(G(N))$ in their bosonic Fock-space representation. For such a purpose, as in the $p = 2$ PSSQM case (Quesne and Vansteenkiste, 1998), we shall start by assuming

$$Q = \sum_{\nu=0}^{2} (\xi_\nu a + \eta_\nu a^\dagger) P_\nu \qquad (5.4)$$

$$\mathcal{H} = H_0 + \frac{1}{2} \sum_{\nu=0}^{2} r_\nu P_\nu \qquad (5.5)$$

where H_0 is the bosonic oscillator Hamiltonian (3.37) associated with $\mathcal{A}^{(3)}(G(N))$, ξ_ν, η_ν are some complex constants, and r_ν are some real ones, to be selected in such a way that Eqs. (5.1)–(5.3) are satisfied.

Inserting the expression of Q given in Eq. (5.4) into the first condition (5.1), we obtain some restrictions on the parameters ξ_ν, η_ν, leading to two sets of three independent solutions for Q. The solutions belonging to the first set are given by

$$Q = (\xi_{\mu+2} a + \eta_{\mu+2} a^\dagger) P_{\mu+2} \qquad (5.6)$$

where μ takes some fixed, arbitrary value in the set $\{0, 1, 2\}$. Those belonging to the second set can be written as

$$Q' = \xi_{\mu+2} a P_{\mu+2} + \eta_\mu a^\dagger P_\mu \qquad (5.7)$$

and can be obtained from the former by interchanging the roles of Q and $Q^\dagger$ (and changing the μ value). They will be omitted here, since Q and $Q^\dagger$ play a symmetrical role in the pseudoSSQM algebra (5.1)–(5.3).

Considering next the second and third conditions in Eqs. (5.2) and (5.3), with Q given by Eq. (5.6) for some μ value and the corresponding $\mathcal{H}$ given by Eq. (5.5), we get the restrictions

$$\xi_{\mu+2}(-2 + \alpha_\mu + r_{\mu+1} - r_{\mu+2}) = 0 \qquad (5.8)$$

$$\eta_{\mu+2}(2 - \alpha_{\mu+1} + r_\mu - r_{\mu+2}) = 0 \qquad (5.9)$$

and

$$(|\xi_{\mu+2}|^2 + |\eta_{\mu+2}|^2)\xi_{\mu+2} = 4c^2\xi_{\mu+2} \qquad (5.10)$$

$$(|\xi_{\mu+2}|^2 + |\eta_{\mu+2}|^2)\,\eta_{\mu+2} = 4c^2\eta_{\mu+2} \qquad (5.11)$$

$$\xi_{\mu+2}[(|\xi_{\mu+2}|^2 + |\eta_{\mu+2}|^2)(1 + \alpha_{\mu+1}) + |\eta_{\mu+2}|^2(1 + \alpha_{\mu+2})]$$

$$= 2c^2\xi_{\mu+2}(3 + 2\alpha_{\mu+1} + \alpha_{\mu+2} + r_{\mu+2}) \tag{5.12}$$

$$\eta_{\mu+2}|\eta_{\mu+2}|^2(1 + \alpha_{\mu+2}) = 2c^2\eta_{\mu+2}(1 + \alpha_{\mu+2} + r_{\mu+2}) \tag{5.13}$$

respectively.

Equations (5.8) and (5.9) have three independent solutions:

$$\xi_{\mu+2} \neq 0, \qquad \eta_{\mu+2} \neq 0,$$
$$r_{\mu+1} = 2 - \alpha_\mu + r_{\mu+2}, \qquad r_\mu = -2 + \alpha_{\mu+1} + r_{\mu+2} \tag{5.14}$$

$$\xi_{\mu+2} \neq 0, \qquad \eta_{\mu+2} = 0, \qquad r_{\mu+1} = 2 - \alpha_\mu + r_{\mu+2} \tag{5.15}$$

$$\xi_{\mu+2} = 0, \qquad \eta_{\mu+2} \neq 0, \qquad r_\mu = -2 + \alpha_{\mu+1} + r_{\mu+2} \tag{5.16}$$

Since the third solution can be obtained from the second one by substituting $Q^\dagger$ for Q, and changing the μ value, we are only left with the first two solutions (5.14) and (5.15).

Introducing Eq. (5.14) into Eqs. (5.10)–(5.13), we get the additional conditions

$$|\xi_{\mu+2}| = \sqrt{4c^2 - |\eta_{\mu+2}|^2},$$
$$r_{\mu+2} = \frac{1}{2c^2}(1 + \alpha_{\mu+2})(|\eta_{\mu+2}|^2 - 2c^2) \tag{5.17}$$

which define with Eq. (5.14) the first set of solutions of the pseudoSSQM algebra (5.1)–(5.3). As we can fix the overall, arbitrary phase of Q in such a way that $\eta_{\mu+2}$ is real and positive, we obtain for each μ value a two-parameter family of operators

$$Q(\eta_{\mu+2}, \varphi) = (\eta_{\mu+2}a^\dagger + e^{i\varphi}\sqrt{4c^2 - \eta_{\mu+2}^2}a)P_{\mu+2} \tag{5.18}$$

$$\mathcal{H}(\eta_{\mu+2}) = N + \tfrac{1}{2}(2\gamma_{\mu+2} + r_{\mu+2} - 1)I + 2P_{\mu+1} + P_{\mu+2} \tag{5.19}$$

where $0 < \eta_{\mu+2} < 2|c|$, $0 \leq \varphi < 2\pi$, and $r_{\mu+2}$ is given by Eq. (5.17). If we choose, for instance, $\eta_{\mu+2} = \sqrt{2}|c|$ and $\varphi = 0$, we get $r_{\mu+2} = 0$, and

$$Q = c\sqrt{2}(a^\dagger + a)P_{\mu+2} \tag{5.20}$$

$$\mathcal{H} = N + \tfrac{1}{2}(2\gamma_{\mu+2} - 1)I + 2P_{\mu+1} + P_{\mu+2} \tag{5.21}$$

Note that this choice does not change $\mathcal{H}$ in any significant way since it only produces an overall shift of its spectrum.

Introducing now Eq. (5.15) into Eqs. (5.10)–(5.13), we get instead the additional conditions

$$|\xi_{\mu+2}| = 2|c|, \qquad r_{\mu+2} = -1 - \alpha_{\mu+2} \qquad (5.22)$$

which define with Eq. (5.15) a second set of solutions of the pseudoSSQM algebra (5.1)–(5.3). Choosing this time the overall, arbitrary phase of Q in such a way that $\xi_{\mu+2}$ is real and positive, we obtain for each μ value a one-parameter family of operators

$$Q = 2|c|aP_{\mu+2} \qquad (5.23)$$

$$\mathcal{H}(r_\mu) = N + \tfrac{1}{2}(2\gamma_{\mu+2} - \alpha_{\mu+2})I + \tfrac{1}{2}(1 - \alpha_{\mu+1} + \alpha_{\mu+2} + r_\mu)P_\mu$$

$$+ P_{\mu+1} \qquad (5.24)$$

where the parameter r_μ does change the Hamiltonian spectrum in a significant way.

The pseudosupersymmetric Hamiltonian, corresponding to the first solution (5.20), (5.21), coincides with the $p = 2$ parasupersymmetric Hamiltonian previously obtained (Quesne and Vansteenkiste, 1998), and defined for arbitrary p in Eq. (4.34) of the present work (but the respective charges are of course different). Its spectrum and its ground-state energy are therefore given by Eqs. (4.35), (4.36), and by Eq. (4.37), respectively.

On the contrary, the pseudosupersymmetric Hamiltonian $\mathcal{H}(r_\mu)$, corresponding to the second solution (5.23), (5.24), is new, and its spectrum is given by

$$\mathcal{E}_{3k+\nu} = 3k + \tfrac{1}{2}(2\gamma_{\mu+2} - \alpha_{\mu+2} + 2\mu - 2)$$

$$\text{if} \quad \nu = 0, 1, \ldots, \mu - 1 \qquad (5.25)$$

$$\mathcal{E}_{3k+\mu} = 3k + \tfrac{1}{2}(2\gamma_\mu + r_\mu + 2\mu + 1) \qquad (5.26)$$

$$\mathcal{E}_{3k+\nu} = 3k + \tfrac{1}{2}(2\gamma_{\mu+2} - \alpha_{\mu+2} + 2\mu + 4)$$

$$\text{if} \quad \nu = \mu + 1, \mu + 2, \ldots, 2 \qquad (5.27)$$

Its levels are therefore equally spaced only if $r_\mu = (\alpha_{\mu+1} - \alpha_{\mu+2} + 3)$ mod 6. If r_μ is small enough, the ground state is nondegenerate, and its energy is negative for $\mu = 1$, or may have any sign for $\mu = 0$ or 2. On the contrary, if r_μ is large enough, the ground state remains nondegenerate with a vanishing energy in the former case, while it becomes twofold degenerate with a positive energy in the latter. For some intermediate r_μ value, one gets a two- or threefold-degenerate ground state with a vanishing or positive energy, respectively.

6. APPLICATION OF C_3-EXTENDED OSCILLATOR ALGEBRAS TO OSSQM OF ORDER TWO

OSSQM of arbitrary order p was developed by Khare *et al.* (1993a) by combining standard bosons with orthofermions of order p. The latter had

been previously introduced by Mishra and Rajasekaran (1991a, b), by replacing Pauli's exclusion principle by a new, more stringent one. OSSQM is formulated in terms of an orthosupersymmetric Hamiltonian $\mathscr{H}$, and p orthosupercharge operators Q_r, $Q_r^\dagger$, $r = 1, 2, \ldots, p$, satisfying the relations

$$Q_r Q_s = 0 \tag{6.1}$$

$$[\mathscr{H}, Q_r] = 0 \tag{6.2}$$

$$Q_r Q_s^\dagger + \delta_{r,s} \sum_{t=1}^{p} Q_t^\dagger Q_t = 2\delta_{r,s}\mathscr{H} \tag{6.3}$$

and their Hermitian conjugates, where r and s run over $1, 2, \ldots, p$.

We plan to show that for $p = 2$, the OSSQM algebra (6.1)–(6.3) can be realized in terms of the generators of $\mathscr{A}^{(3)}(G(N))$ in their bosonic Fock-space representation. For such a purpose, let us set

$$Q_1 = \sum_{\nu=0}^{2} (\xi_\nu a + \eta_\nu a^\dagger)P_\nu \tag{6.4}$$

$$Q_2 = \sum_{\nu=0}^{2} (\zeta_\nu a + \rho_\nu a^\dagger)P_\nu \tag{6.5}$$

$$\mathscr{H} = H_0 + \frac{1}{2} \sum_{\nu=0}^{2} r_\nu P_\nu \tag{6.6}$$

where we now have at our disposal four types of complex constants ξ_ν, η_ν, ζ_ν, ρ_ν and one type of real ones r_ν to adjust in order that Eqs. (6.1)–(6.3) be satisfied.

Let us first consider Eq. (6.1) for $r = s = 1, 2$. From the study carried out in Section 5, we know that for each r in the set $\{1, 2\}$, the equation $Q_r^2 = 0$ admits two different types of solutions, given in Eqs. (5.6) and (5.7), respectively, and connected by the symmetry $Q \leftrightarrow Q^\dagger$. In the present case, we have to distinguish them, since the OSSQM algebra (6.1)–(6.3) is not invariant under such a symmetry. Hence, for the couple of orthosupersymmetric charges (Q_1, Q_2) we get seven types of solutions of $Q_1^2 = Q_2^2 = 0$, namely Q_1 and Q_2 may be both of type Q or Q' with the same or adjacent μ values, or Q_1 is of type Q corresponding to a given μ value, and Q_2 of type Q' corresponding to μ, $\mu + 1$, or $\mu + 2$. Here, we take into account the fact that the algebra (6.1)–(6.3) is invariant under the exchange $Q_1 \leftrightarrow Q_2$.

Imposing next Eqs. (6.1) and (6.3) for $r \neq s$, i.e., $Q_1 Q_2 = Q_2 Q_1 = Q_1 Q_2^\dagger = 0$, we obtain that those seven cases for (Q_1, Q_2) actually reduce to two, given by

$$Q_1 = \xi_{\mu+2}aP_{\mu+2} + \eta_\mu a^\dagger P_\mu, \qquad Q_2 = \zeta_{\mu+2}aP_{\mu+2} + \rho_\mu a^\dagger P_\mu \qquad (6.7)$$

and

$$Q_1 = \xi_{\mu+2}aP_{\mu+2}, \qquad Q_2 = \rho_\mu a^\dagger P_\mu \qquad (6.8)$$

respectively, where for the first one, we have the additional conditions

$$\xi_{\mu+2}\zeta^*_{\mu+2} + \eta_\mu\rho^*_\mu = 0 \qquad (\xi_{\mu+2}, \eta_\mu \neq 0) \qquad (6.9)$$

$$\alpha_{\mu+1} = -1 \qquad (6.10)$$

Note that the latter is compatible with conditions (3.34) for the existence of the bosonic Fock-space representation only for $\mu = 0$ and $\mu = 1$.

Equation (6.2) now leads to the same conditions for both choices (6.7) and (6.8), namely

$$r_\mu = 4 + \alpha_{\mu+1} + r_{\mu+2}, \qquad r_{\mu+1} = 2 - \alpha_\mu + r_{\mu+2} \qquad (6.11)$$

It only remains to impose Eq. (6.3) for $r = s = 1, 2$. For the first couple of operators (Q_1, Q_2) given in Eqs. (6.7), (6.9), and (6.10), we obtain the additional restrictions

$$|\xi_{\mu+2}|^2 + |\eta_\mu|^2 = 2, \qquad |\zeta_{\mu+2}|^2 = |\eta_\mu|^2,$$
$$|\rho_\mu|^2 = |\xi_{\mu+2}|^2, \qquad \xi_{\mu+2}\eta^*_\mu + \zeta_{\mu+2}\rho^*_\mu = 0 \qquad (6.12)$$

and

$$r_\mu = 1 + \alpha_\mu, \qquad r_{\mu+1} = 0, \qquad r_{\mu+2} = -1 - \alpha_{\mu+2} \qquad (6.13)$$

Combining Eqs. (6.9) and (6.12), we get

$$\xi_{\mu+2} = |\xi_{\mu+2}|e^{i\alpha}, \qquad\qquad \eta_\mu = \sqrt{2 - |\xi_{\mu+2}|^2}\, e^{i\beta} \qquad (6.14)$$

$$\zeta_{\mu+2} = -\sqrt{2 - |\xi_{\mu+2}|^2}\, e^{i(\alpha-\beta+\gamma)}, \qquad \rho_\mu = |\xi_{\mu+2}|\, e^{i\gamma} \qquad (6.15)$$

where $0 < |\xi_{\mu+2}| < \sqrt{2}$, and $0 \leq \alpha, \beta, \gamma < 2\pi$. In addition, we find that Eqs. (6.10), (6.11), and (6.13) are compatible, and can be combined into the relations

$$r_\mu = 1 + \alpha_\mu, \qquad r_{\mu+1} = 0, \qquad r_{\mu+2} = -2 + \alpha_\mu, \qquad \alpha_{\mu+1} = -1 \qquad (6.16)$$

Choosing the overall, arbitrary phases of Q_1 and Q_2 in such a way that $\xi_{\mu+2}$ and ρ_μ are real and positive, and setting $\beta = \varphi$, we obtain for $\mu = 0$ or 1 a two-parameter family of solutions of Eqs. (6.1)–(6.3),

$$Q_1(\xi_{\mu+2}, \varphi) = \xi_{\mu+2}aP_{\mu+2} + e^{i\varphi}\sqrt{2 - \xi^2_{\mu+2}}\, a^\dagger P_\mu \qquad (6.17)$$

$$Q_2(\xi_{\mu+2}, \varphi) = -e^{-i\varphi}\sqrt{2 - \xi^2_{\mu+2}}\, aP_{\mu+2} + \xi_{\mu+2}a^\dagger P_\mu \qquad (6.18)$$

$$\mathcal{H} = N + \tfrac{1}{2}(2\gamma_{\mu+1} - 1)\, I + 2P_\mu + P_{\mu+1} \qquad (6.19)$$

where $0 < \xi_{\mu+2} < \sqrt{2}$, $0 \leq \varphi < 2\pi$, and $\alpha_{\mu+1} = -1$.

For the second couple of operators (Q_1, Q_2) given in Eq. (6.8), Eq. (6.3) with $r = s = 1$, 2 leads to the conditions

$$|\xi_{\mu+2}|^2 = |\rho_\mu|^2 = 2 \qquad (6.20)$$

and to Eqs. (6.10) and (6.13). Hence, with an appropriate choice of phases, we obtain Eqs. (6.17)–(6.19) with $\xi_{\mu+2} = \sqrt{2}$. We conclude that the most general solution of the OSSQM algebra (6.1)–(6.3) that can be written in the form (6.4)–(6.6) is given by Eqs. (6.17)–(6.19), where $\mu \in \{0, 1\}$, $0 < \xi_{\mu+2} \leq \sqrt{2}$, $0 \leq \varphi < 2\pi$, and $\alpha_{\mu+1} = -1$.

The orthosupersymmetric Hamiltonian $\mathcal{H}$ in Eq. (6.19) is independent of the parameters $\xi_{\mu+2}$, φ. All the levels of its spectrum are equally spaced. For $\mu = 0$, they are threefold degenerate, since

$$\mathcal{E}_{3k} = \mathcal{E}_{3k+1} = \mathcal{E}_{3k+2} = 3k + \tfrac{1}{2}(2\gamma_1 + 3) \qquad (6.21)$$

OSSQM is therefore broken, and the ground-state energy

$$\mathcal{E}_0 = \mathcal{E}_1 = \mathcal{E}_2 = \tfrac{1}{2}(2\gamma_1 + 3) = \alpha_0 + 1 \qquad (6.22)$$

is positive. On the contrary, for $\mu = 1$, only the excited states are threefold degenerate, since

$$\mathcal{E}_{3(k+1)} = \mathcal{E}_{3k+1} = \mathcal{E}_{3k+2} = 3k + \tfrac{1}{2}(2\gamma_2 + 5) \qquad (6.23)$$

OSSQM is then unbroken, and the ground-state energy

$$\mathcal{E}_0 = \tfrac{1}{2}(2\gamma_2 - 1) = -\tfrac{1}{2}(\alpha_2 + 1) \qquad (6.24)$$

vanishes. Such results agree with the general conclusions of Khare *et al.* (1993a).

For p values greater than two, the OSSQM algebra (6.1)–(6.3) becomes rather complicated because the number of equations to be fulfilled increases considerably. A glance at the 18 independent conditions for $p = 3$ led us to the conclusion that the $\mathcal{A}^{(4)}(G(N))$ algebra is not rich enough to contain operators satisfying Eqs. (6.1)–(6.3). Contrary to what happens for PSSQM, for OSSQM the $p = 2$ case is therefore not representative of the general one.

7. SOME DEFORMED C_λ-EXTENDED OSCILLATOR ALGEBRAS

The purpose of the present section is to construct some deformations of the C_λ-extended oscillator algebras $\mathcal{A}^{(\lambda)}_{\alpha_0\alpha_1\ldots\alpha_{\lambda-2}}$ subject to the condition

that they admit three Casimir operators analogous to $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$ defined in Eqs. (2.10)–(2.12).

Let us consider a class of algebras generated by I, $a^\dagger$, $a = (a^\dagger)^\dagger$, $N = N^\dagger$, $P_\mu = P_\mu^\dagger$, $\mu = 0, 1, \ldots, \lambda - 1$, satisfying the defining relations (2.5)–(2.7) of $\mathscr{A}_{\alpha_0\alpha_2\ldots\alpha_{\lambda-2}}^{(\lambda)}$ except for the commutator of a and $a^\dagger$ in Eq. (2.7), which is replaced by the quommutator (or q-deformed commutator)

$$[a, a^\dagger]_q \equiv aa^\dagger - qa^\dagger a = H(N) + K(N) \sum_{\mu=0}^{\lambda-1} \alpha_\mu P_\mu \qquad (7.1)$$

where $q \in \mathbb{R}^+$, $\alpha_\mu \in \mathbb{R}$, and $H(N)$, $K(N)$ are some real, analytic functions of N.

The operators $\mathscr{C}_1$, $\mathscr{C}_2$ of Eqs. (2.10), (2.11) remain invariants of the new algebras. We will determine the constraints that the existence of a third Casimir operator of the type

$$\tilde{\mathscr{C}}_3 = q^{-N}\left(D(N) + E(N) \sum_{\mu=0}^{\lambda-1} \beta_\mu P_\mu - a^\dagger a \right) \qquad (7.2)$$

imposes on $H(N)$ and $K(N)$, assuming that Eq. (2.9) is the only relation satisfied by the α_μ. Here β_μ, $\mu = 0, 1, \ldots, \lambda - 1$, and $D(N)$, $E(N)$ are assumed to be some real constants and some real, analytic functions of N, respectively. In the case of the undeformed algebras $\mathscr{A}_{\alpha_0\alpha_1\ldots\alpha_{\lambda-2}}^{(\lambda)}$, one has $q = 1$, $H(N) = K(N) = I$, and $\tilde{\mathscr{C}}_3$ reduces to $\mathscr{C}_3$ given in Eq. (2.12), with $D(N) = N$, $E(N) = I$, and β_μ defined by Eq. (2.13) in terms of the α_μ.

In the realization (2.16), the deformed algebras defined by Eqs. (2.5), (2.6), (2.9), and (7.1) reduce to GDOAs $\mathscr{A}_q^{(\lambda)}(G(N))$ with $q \neq 1$ and $G(N)$ given by the right-hand side of Eq. (7.1). Then $\tilde{\mathscr{C}}_3$ reduces to the standard Casimir operator $\tilde{\mathscr{C}}$ of such algebras, and $F(N) = D(N) + E(N) \Sigma_{\mu=0}^{\lambda-1} \beta_\mu P_\mu$ becomes the GDOA structure function, satisfying the equation $F(N + 1) - qF(N) = G(N)$ (Katriel and Quesne, 1996; Quesne and Vansteenkiste, 1996, 1997).

Going back to the general case, we note that since $\tilde{\mathscr{C}}_3$ is a Hermitian operator commuting with N and P_μ, we only have to impose the condition $[\tilde{\mathscr{C}}_3, a] = 0$. By using the defining relations, it is easy to show that the latter is equivalent to the two functional equations

$$D(N + 1) - qD(N) = H(N) \qquad (7.3)$$

$$E(N + 1)\beta_{\mu+1} - qE(N)\beta_\mu = K(N)\alpha_\mu, \qquad \mu = 0, 1, \ldots, \lambda - 1 \quad (7.4)$$

where we assume as usual $\beta_\lambda = \beta_0$. Equation (7.3) is similar to the equation appearing in the construction of $\tilde{\mathscr{C}}$ for GDOAs with $q \neq 1$ (Katriel and Quesne, 1996; Quesne and Vansteenkiste, 1996, 1997), while Eq. (7.4) is a new functional equation, whose solutions will now be determined.

For such a purpose, let us consider the following nonhomogeneous system of λ linear equations in λ unknowns β_μ, $\mu = 0, 1, \ldots, \lambda - 1$,

$$-qE(x)\beta_\mu + E(x + 1)\beta_{\mu+1} = K(x)\alpha_\mu, \qquad \mu = 0, 1, \ldots, \lambda - 1 \quad (7.5)$$

$$\beta_\lambda \equiv \beta_0 \tag{7.6}$$

where x is some real variable.

If the determinant of its coefficient matrix is nonvanishing, i.e., if

$$[E(x + 1)]^\lambda - [qE(x)]^\lambda \neq 0 \tag{7.7}$$

or, equivalently,

$$E(x) \neq bq^x \tag{7.8}$$

and

$$E(x) \neq b'(-q)^x \qquad \text{if} \quad \lambda \text{ is even} \tag{7.9}$$

where b, b' are some real, nonvanishing constants, then the system has one and only one solution, given by

$$\beta_\mu = \frac{[qE(x)]^{\lambda-1}K(x)}{[E(x + 1)]^\lambda - [qE(x)]^\lambda} \sum_{\nu=0}^{\lambda-1} \left(\frac{E(x + 1)}{qE(x)}\right)^\nu \alpha_{\mu+\nu},$$

$$\mu = 0, 1, \ldots, \lambda - 1 \tag{7.10}$$

Since, by definition, β_μ, $\mu = 0, 1, \ldots, \lambda - 1$, are constants, the functions $E(x)$ and $K(x)$ should be chosen in such a way that the dependence on x disappears on the right-hand side of Eq. (7.10).

Let us first consider β_0. By using Eq. (2.9) to express α_0 in terms of $\alpha_1, \alpha_2, \ldots, \alpha_{\lambda-1}$, we can rewrite β_0 as

$$\beta_0 = \frac{[qE(x)]^{\lambda-1}K(x)}{[E(x + 1)]^\lambda - [qE(x)]^\lambda} \sum_{\nu=1}^{\lambda-1} \left[\left(\frac{E(x + 1)}{qE(x)}\right)^\nu - 1\right]\alpha_\nu \tag{7.11}$$

Since $\alpha_1, \alpha_2, \ldots, \alpha_{\lambda-1}$ are assumed to be independent, the coefficient of each of them on the right-hand side of Eq. (7.11) should reduce to some real constant, which we denote by e_ν, $\nu = 1, 2, \ldots, \lambda - 1$. Hence we get the system of equations

$$\frac{1}{e_1}\left(\frac{E(x + 1)}{qE(x)} - 1\right) = \frac{[E(x + 1)]^\lambda - [qE(x)]^\lambda}{[qE(x)]^{\lambda-1}K(x)} \tag{7.12}$$

$$\frac{1}{e_1}\left(\frac{E(x + 1)}{qE(x)} - 1\right) = \frac{1}{e_\nu}\left[\left(\frac{E(x + 1)}{qE(x)}\right)^\nu - 1\right],$$

$$\nu = 2, 3, \ldots, \lambda - 1 \tag{7.13}$$

to determine the constraints on $E(x)$ and $K(x)$.

For $\lambda = 2$, we are only left with the first equation (7.12), yielding the constraint

$$K(x) = e_1[E(x + 1) + qE(x)] \tag{7.14}$$

Introducing the latter into Eq. (7.10) and using Eq. (2.9) again, we obtain

$$\beta_\mu = -e_1\alpha_\mu, \qquad \mu = 0, 1 \tag{7.15}$$

which are constants, as they should be. Incorporating the constant e_1 into the $E(x)$ definition, we conclude that the algebras defined by Eqs. (2.5), (2.6) with $\lambda = 2$ and

$$[a, a^\dagger]_q = H(N) + [E(N + 1) + qE(N)](\alpha_0 P_0 + \alpha_1 P_1) \tag{7.16}$$

where α_0, α_1 satisfy Eq. (2.9), $H(N)$ is arbitrary, and $E(N) \neq (\pm q)^N$, admit the three Casimir operators (2.10), (2.11), and

$$\tilde{\mathscr{C}}_3 = q^{-N}[D(N) - E(N)(\alpha_0 P_0 + \alpha_1 P_1) - a^\dagger a] \tag{7.17}$$

where $D(N)$ is some solution of Eq. (7.3). By choosing that solution for which $D(0) = \alpha_0 E(0)$, $\tilde{\mathscr{C}}_3$ vanishes in the bosonic Fock-space representation.

For $\lambda > 2$, Eq. (7.13) for $\nu = 2$ yields the constraint

$$E(x + 1) = \left(\frac{e_2}{e_1} - 1\right) qE(x) \tag{7.18}$$

whose solution is given by

$$E(x) = bk^x \tag{7.19}$$

where b is some real constant and $k \equiv (e_1^{-1} e_2 - 1)q$. From Eqs. (7.8) and (7.9), it follows that for any λ, $k \neq q$, and in addition for even λ, $k \neq -q$. Equation (7.12) then provides the expression of $K(x)$,

$$K(x) = Bk^x \tag{7.20}$$

where $B \equiv e_1 bq^{2-\lambda} (k^\lambda - q^\lambda)/(k - q)$, while for the remaining ν values, Eq. (7.13) leads to the conditions

$$e_\nu = e_1 q^{1-\nu} \frac{k^\nu - q^\nu}{k - q}, \qquad \nu = 2, 3, \ldots, \lambda - 1 \tag{7.21}$$

Hence, from Eq. (7.10), β_μ is given by

$$\beta_\mu = \frac{Bq^{\lambda-1}}{b(k^\lambda - q^\lambda)} \sum_{\nu=0}^{\lambda-1} \left(\frac{k}{q}\right)^\nu \alpha_{\mu+\nu}, \qquad \mu = 0, 1, \ldots, \lambda - 1 \quad (7.22)$$

and therefore reduces to some constant, as it should. We conclude that for $\lambda > 2$, the algebras defined by Eqs. (2.5), (2.6), and

$$[a, a^\dagger]_q = H(N) + Bk^N \sum_{\mu=0}^{\lambda-1} \alpha_\mu P_\mu \qquad (7.23)$$

where $H(N)$ and B are arbitrary, α_μ satisfies Eq. (2.9), $k \neq q$ for any λ, and $k \neq -q$ for even λ, admit the three Casimir operators (2.10), (2.11), and

$$\tilde{\mathcal{C}}_3 = q^{-N} \left\{ D(N) + \frac{Bq^{\lambda-1}}{k^\lambda - q^\lambda} k^N \sum_{\mu=0}^{\lambda-1} \left[\sum_{\nu=0}^{\lambda-1} \left(\frac{k}{q}\right)^\nu \alpha_{\mu+\nu} \right] P_\mu - a^\dagger a \right\} \qquad (7.24)$$

where $D(N)$ is some solution of Eq. (7.3). By choosing that solution for which $D(0) = -Bq^{\lambda-1}(k^\lambda - q^\lambda)^{-1} \sum_{\nu=0}^{\lambda-1} (k/q)^\nu \alpha_\nu$, $\tilde{\mathcal{C}}_3$ vanishes in the bosonic Fock-space representation.

It remains to consider the cases where the coefficient matrix of system (7.5), (7.6) has a vanishing determinant. If $E(x) = bq^x$, where b is some real constant, then Eqs. (7.5) and (7.6) become

$$-\beta_\mu + \beta_{\mu+1} = (bq)^{-1} \frac{K(x)}{q^x} \alpha_\mu, \qquad \mu = 0, 1, \ldots, \lambda - 1 \quad (7.25)$$

$$\beta_\lambda \equiv \beta_0 \qquad (7.26)$$

Since the β_μ are constants, we obtain

$$K(x) = Bq^x \qquad (7.27)$$

where B is some real constant, and therefore

$$\beta_\mu = \frac{B}{bq} \sum_{\nu=0}^{\mu-1} \alpha_\nu + \beta_0, \qquad \mu = 1, 2, \ldots, \lambda - 1 \qquad (7.28)$$

We conclude that the algebras defined by Eqs. (2.5), (2.6), and

$$[a, a^\dagger]_q = H(N) + Bq^N \sum_{\mu=0}^{\lambda-1} \alpha_\mu P_\mu \qquad (7.29)$$

where $H(N)$ and B are arbitrary, and α_μ satisfies Eq. (2.9), admit the three Casimir operators (2.10), (2.11), and

$$\tilde{\mathcal{C}}_3 = q^{-N} \left[D(N) + Bq^{N-1} \sum_{\mu=1}^{\lambda-1} \left(\sum_{\nu=0}^{\mu-1} \alpha_\nu \right) P_\mu - a^\dagger a \right] \qquad (7.30)$$

where we have set $\beta_0 = 0$ (thereby eliminating a multiple of the unit operator),

and $D(N)$ is some solution of Eq. (7.3). By choosing that solution for which $D(0) = 0$, $\tilde{\mathscr{C}}_3$ vanishes in the bosonic Fock-space representation.

Finally, if λ is even and $E(x) = b(-q)^x$, where b is some real constant, then Eqs. (7.5) and (7.6) become

$$\beta_\mu + \beta_{\mu+1} = -(bq)^{-1} \frac{K(x)}{(-q)^x} \alpha_\mu, \qquad \mu = 0, 1, \ldots, \lambda - 1 \quad (7.31)$$

$$\beta_\lambda \equiv \beta_0 \tag{7.32}$$

The β_μ constancy implies again that

$$K(x) = B(-q)^x \tag{7.33}$$

where B is some real constant. Equation (7.31) is then equivalent to

$$\beta_0 + \beta_1 = -\frac{B}{bq} \alpha_0 \tag{7.34}$$

$$\beta_{\mu+2} - \beta_\mu = -\frac{B}{bq} (\alpha_{\mu+1} - \alpha_\mu), \qquad \mu = 0, 1, \ldots, \lambda - 2 \quad (7.35)$$

The solution of Eqs. (7.34) and (7.35) is given by

$$\beta_\mu = -\frac{B}{bq} \left(\sum_{\nu=0}^{(\mu-2)/2} \alpha_{2\nu+1} - \sum_{\nu=0}^{(\mu-2)/2} \alpha_{2\nu} \right) + \beta_0, \qquad \text{if} \quad \mu \text{ is even} \quad (7.36)$$

$$\beta_\mu = -\frac{B}{bq} \left(\sum_{\nu=0}^{(\mu-1)/2} \alpha_{2\nu} - \sum_{\nu=0}^{(\mu-3)/2} \alpha_{2\nu+1} \right) - \beta_0 \qquad \text{if} \quad \mu \text{ is odd} \quad (7.37)$$

Condition (7.32) is consistent with Eq. (7.36) if and only if we impose that $\sum_{\nu=0}^{(\lambda-2)/2} \alpha_{2\nu+1} = \sum_{\nu=0}^{(\lambda-2)/2} \alpha_{2\nu}$, or by taking Eq. (2.9) into account, $\sum_{\nu=0}^{(\lambda-2)/2} \alpha_{2\nu} = 0$. Since we have assumed that the α_μ do not satisfy any extra relation apart from Eq. (2.9), the case $E(x) = b(-q)^x$ has to be rejected.

We have therefore found three deformed C_λ-extended oscillator algebras admitting three Casimir operators $\mathscr{C}_1$, $\mathscr{C}_2$, $\tilde{\mathscr{C}}_3$. They correspond to Eqs. (7.16) and (7.17), (7.23) and (7.24), and (7.29) and (7.30), respectively.

The deformed Calogero–Vasiliev algebra introduced by Brzeziński *et al.* (1993), for which

$$[a, a^\dagger]_q = q^{-N}(1 + 2\alpha K), \qquad K = (-1)^N \tag{7.38}$$

is a special case of Eq. (7.16), corresponding to

$$H(N) = q^{-N}, \qquad E(N) = \frac{2q^{-N}}{q + q^{-1}}, \qquad \alpha_0 = -\alpha_1 = \alpha \quad (7.39)$$

From Eq. (7.3), we obtain

$$D(N) = \frac{q^N - q^{-N}}{q - q^{-1}} + \frac{2\alpha q^N}{q + q^{-1}} \tag{7.40}$$

so that the Casimir operator (7.17) becomes

$$\tilde{\mathscr{C}}_3 = q^{-N} \left(\frac{q^N - q^{-N}}{q - q^{-1}} + \frac{2\alpha(q^N - q^{-N} K)}{q + q^{-1}} - a^\dagger a \right) \tag{7.41}$$

In a given unirrep whose basis states are given by Eq. (3.1) and satisfy relations similar to Eqs. (3.2) and (3.3) with $\mathscr{C} = \mathscr{C}_3$ replaced by $\tilde{\mathscr{C}}_3$, we obtain from Eq. (7.41) that λ_n can be expressed as

$$\lambda_n = -q^{n_0+n}c + \frac{q^{n_0+n} - q^{-n_0-n}}{q - q^{-1}} + 2\alpha \frac{q^{n_0+n} - (-q)^{-n_0-n}}{q + q^{-1}} \tag{7.42}$$

or

$$\lambda_n = q^n \lambda_0 + q^{-n_0} \left(\frac{q^n - q^{-n}}{q - q^{-1}} + B \frac{q^n - (-q)^{-n}}{q + q^{-1}} \right), \qquad B \equiv 2\alpha(-1)^{n_0} \tag{7.43}$$

This equation is consistent with Eq. (14) of Kosiński *et al.* (1997), wherein the representations of the deformed Calogero–Vasiliev algebra were studied. Note that this result holds although there are some slight discrepancies in the algebra definition between Kosiński *et al.* (1997) and the present work, and the Casimir operator $\tilde{\mathscr{C}}_3$ was not considered in the former.

Some interesting special cases of the algebras corresponding to Eqs. (7.23) and (7.29) are obtained for $q = 1$, $k \neq 1$, $k \neq -1$ (if λ is even) and $k = 1$, $q \neq 1$, $q \neq -1$ (if λ is even) for the former, and $q = 1$ for the latter.

8. CONCLUSION

In the present paper, we studied some mathematical properties of C_λ-extended oscillator algebras $\mathscr{A}^{(\lambda)}_{\alpha_0 \alpha_1 \ldots \alpha_{\lambda-2}}$. We constructed Casimir operators and used them to provide a complete unirrep classification under the assumption that the number operator spectrum is nondegenerate. We established that only BFB and FD unirreps occur, and showed that the unirreps of $\mathscr{A}^{(\lambda)}_{\alpha_0 \alpha_1 \ldots \alpha_{\lambda-2}}$ can be related to those of its GDOA realization $\mathscr{A}^{(\lambda)}(G(N))$.

In addition, we looked for some deformations of $\mathscr{A}^{(\lambda)}_{\alpha_0 \alpha_1 \ldots \alpha_{\lambda-2}}$ subject to the condition that they admit Casimir operators analogous to those of the undeformed algebras. We found three new types of algebras, defined in Eqs. (7.16) and (7.17), (7.23) and (7.24), and (7.29) and (7.30), respectively. The first one includes the Brzeziński *et al.* (1993) deformation of the Calogero–

Vasiliev algebra (Vasiliev, 1991; Polychronakos, 1992; Brink *et al.*, 1992; Brink and Vasiliev, 1993) as a special case.

Furthermore, we established that the bosonic Fock-space realization of $\mathcal{A}^{(\lambda)}(G(N))$ yields a convenient bosonization of several SSQM variants: PSSQM of order $p = \lambda - 1$ for any λ, as well as pseudoSSQM, and OSSQM of order two for $\lambda = 3$. In the former case, we provided a full analysis of the problem, including the construction of the p independent conserved parasupercharges and p bosonic constants admitted by the parasupersymmetric Hamiltonian. Such results generalize those already known for standard SSQM (Brzeziński *et al.*, 1993; Plyushchay, 1996a, b). In the OSSQM case, however, it was not possible to extend the results to p values greater than two in the C_λ-extended oscillator algebra context.

There remain some interesting open questions for future study. Apart from those mentioned in Section 1, we mention two of them here. The first one is to further study deformations both from theoretical and applied viewpoints. Generalizing, for instance, the Macfarlane (1994) deformation of the Calogero–Vasiliev algebra would be an interesting topic. The second issue is to construct some GDOA whose structure would be rich enough to enable the OSSQM bosonization to be carried out for $p > 2$.

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